Tensor Products of Approximately Cohen-Macaulay Rings[*]

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Abstract

Our aim in this article is to study a problem originally raised by Grothendieck. We show that the approximately Cohen-Macaulay property is preserved for the tensor product of algebras over a field k. We also discuss the converse problem.

Keywords: Approximately Cohen-Macaulay ring, Flat homomorphism of rings, Tensor product.

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1 Introduction

All rings and algebras considered in this article are commutative Noetherian with identity element, and all ring homomorphisms are unital. Throughout, k stands for a field. Let $(R, \mathfrak{m})$ be a local ring with $\dim(R) = d$. Recall that R is a Gorenstein ring if and only if there is an element a of $\mathfrak{m}$ such that R/a^nR is a Gorenstein ring of dimension $d-1$ for every integer $n > 0$ (cf. [7]). Clearly, this is not true for Cohen-Macaulay rings. The local ring R is called an *approximately Cohen-Macaulay* ring if either $\dim(R) = 0$ or there exists an element a of $\mathfrak{m}$ such that R/a^nR is a Cohen-Macaulay ring of dimension $d-1$ for every integer $n > 0$ (cf. [5]). It is shown that if R is an approximately Cohen-Macaulay ring, then so is the ring $R_\mathfrak{p}$ for any prime ideal $\mathfrak{p}$ (see Theorem 2). Therefore, the concept of approximately Cohen-Macaulay is extended to nonlocal rings as follows. A ring R is an *approximately Cohen-Macaulay* ring if for all prime ideals $\mathfrak{p}$ of R, the ring $R_\mathfrak{p}$ is an approximately Cohen-Macaulay ring. It is well known that the tensor product $R \otimes_A S$ of regular rings is not regular in general, even if we assume R and S are A-algebras, where A is a field (cf. [10, Remark 7]). In [11, Remark 1.7], Watanabe, Ishikawa, Tachibana and Otsuka showed that under a suitable condition, tensor products of regular rings are complete intersections. It is proven in [6] that the tensor product $R \otimes_A S$ of Cohen-Macaulay rings are again Cohen-Macaulay, if we assume that R is a flat A-module and S is a finitely generated A-module, and in [11], it is shown that the

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same is true for Gorenstein rings. In [1], Bouchiba and Kabbaj showed that if R and S are k-algebras such that $R \otimes_k S$ is Noetherian, then $R \otimes_k S$ is a Cohen-Macaulay ring if and only if R and S are Cohen-Macaulay rings. Recently, in [10], Tousi and Yassemi showed that if R and S are nonzero k-algebras such that $R \otimes_k S$ is Noetherian, then $R \otimes_k S$ is a locally complete intersection (resp. Gorenstein, Cohen-Macaulay) if and only if R and S are locally complete intersections (resp. Gorenstein, Cohen-Macaulay).

In this article we shall investigate if the approximately Cohen-Macaulay property is conserved under tensor product operations. It is shown that if $\varphi: \ (R, \mathfrak{m}) \longrightarrow (S, \mathfrak{n})$ is a flat local homomorphism and R is not a Cohen-Macaulay ring, then the following are equivalent (see Theorem 6):

(a) R is an approximately Cohen-Macaulay ring and $S/\mathfrak{m}S$ is a Cohen-Macaulay ring.

(b) S is an approximately Cohen-Macaulay ring and $\mathrm{Ass}_S(S/\mathfrak{p}S) = \mathrm{Assh}_S(S/\mathfrak{p}S)$ for every $\mathfrak{p} \in \mathrm{Assh}(R)$.

Further, if R is a homomorphic image of a Cohen-Macaulay local ring, then the next condition is also equivalent:

(c) S is an approximately Cohen-Macaulay ring.

We will also prove the following result. Let R and S be nonzero k-algebras such that $T := R \otimes_k S$ is Noetherian. Assume that R is not a Cohen-Macaulay ring. Then the following hold (see Theorem 10):

(i) If R is an approximately Cohen-Macaulay ring and S is a Cohen-Macaulay ring, then T is an approximately Cohen-Macaulay ring.

(ii) If T is an approximately Cohen-Macaulay ring, then S is a Cohen-Macaulay ring.

(iii) If R is a homomorphic image of a Cohen-Macaulay ring or k is algebraically closed, then the following conditions are equivalent:

(a) T is an approximately Cohen-Macaulay ring.

(b) R is an approximately Cohen-Macaulay ring and S is a Cohen-Macaulay ring.

2 Main Results

For a finitely generated R-module M of finite Krull dimension, recall that

$$\mathrm{Assh}_R(M) = \big\{\mathfrak{p} \in \mathrm{Supp}_R(M) | \dim(R/\mathfrak{p}) = \dim(\mathrm{M})\big\},$$

and denote $\mathrm{Assh}_R(R) = \mathrm{Assh}(R)$. Let $U_R(0) = \bigcap_{\mathfrak{p}\in\mathrm{Assh}(R)} I(\mathfrak{p})$, where $(0) = \bigcap_{\mathfrak{p}\in\mathrm{Ass}(R)} I(\mathfrak{p})$ denotes a minimal primary decomposition of the zero ideal of R (cf. [5]).

Let M be a finitely generated R-module and I an ideal of R such that $IM \neq M$. Then the common length of the maximal M-sequences in I is called the *grade* of I on M, denoted by $\mathrm{grade}_M(I)$. If $(R, \mathfrak{m})$ is a local ring, and M is a finitely generated nonzero R-module, then the grade of $\mathfrak{m}$ on M is called the *depth* of M, denoted by $\mathrm{depth}(M)$.

Theorem 1 (see [5]) *Let R be a local ring with maximal ideal $\mathfrak{m}$ and $\dim(R) = d$. Suppose that R is not a Cohen-Macaulay ring. Then the following conditions are equivalent:*

(i) *R is an approximately Cohen-Macaulay ring.*

(ii) *R contains an ideal I such that I is a Cohen-Macaulay R-module of dimension $d-1$ and R/I is a Cohen-Macaulay ring of dimension d.*

(iii) *$R/U_R(0)$ is a Cohen-Macaulay ring and $\mathrm{depth}(R) = d-1$.*

(iv) (a) $\mathrm{H}^i_{\mathfrak{m}}(R) = (0)$ *for $i \neq d-1, d$.*

(b) $\mathrm{Hom}_R\big(\mathrm{H}^{d-1}_{\mathfrak{m}}(R), \mathrm{E}_R(R/\mathfrak{m})\big)$ *is a Cohen-Macaulay $\hat{R}$-module of dimension $d-1$.*

(c) *The local ring $R/\mathfrak{p}$ is unmixed for every $\mathfrak{p} \in \mathrm{Assh}(R)$, i.e., the equality $\dim(\hat{R}/P) = d$ holds for every $P \in \mathrm{Ass}_{\hat{R}}(\hat{R}/\mathfrak{p}\hat{R})$ and for every $\mathfrak{p} \in \mathrm{Assh}(R)$.*

In this case, the ideal I appearing in assertion (ii) *is uniquely determined and equals $U_R(0)$. Here $\hat{R}$ (resp. $\mathrm{E}_R(R/\mathfrak{m})$) denotes the $\mathfrak{m}$-adic completion of R (resp. the injective hull of $R/\mathfrak{m}$).*

In the following theorem we consider the behavior of approximately Cohen-Macaulay property by passing to localizations.

Theorem 2 *Let $(R, \mathfrak{m})$ be an approximately Cohen-Macaulay ring. Then*

(i) *For any $\mathfrak{p} \in \mathrm{Spec}(R)$, $\dim(R_{\mathfrak{p}}) - \mathrm{depth}(R_{\mathfrak{p}}) \leq 1$.*

(ii) *Suppose R is not a Cohen-Macaulay ring. Then for any $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R_{\mathfrak{p}}$ is not a Cohen-Macaulay ring, $\mathrm{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$.*

(iii) *Suppose R is not a Cohen-Macaulay ring. Then for any $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R_{\mathfrak{p}}$ is not a Cohen-Macaulay ring, $U_{R_{\mathfrak{p}}}(0) = U_R(0)R_{\mathfrak{p}}$.*

(iv) *For any $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ is an approximately Cohen-Macaulay ring.*

Proof. (i): This follows from the fact that $\dim(R_{\mathfrak{p}}) - \mathrm{depth}(R_{\mathfrak{p}}) \leq \dim(R) - \mathrm{depth}(R)$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$ (see [8, Exercise 17.5(ii)]).

(ii): Let $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $R_{\mathfrak{p}}$ is not a Cohen-Macaulay ring. By (i), we have $\operatorname{depth}(R_{\mathfrak{p}}) = \operatorname{ht}(\mathfrak{p}) - 1$. Also, in view of [8, Exercise 17.5(i)], we have

$$\begin{aligned}
\operatorname{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) - 1 &\leq \dim(R) - 1 \\
&= \operatorname{depth}(R) \\
&\leq \operatorname{grade}_R(\mathfrak{p}) + \dim(R/\mathfrak{p}) \\
&\leq \operatorname{depth}(R_{\mathfrak{p}}) + \dim(R/\mathfrak{p}).
\end{aligned}$$

Therefore, $\operatorname{ht}(\mathfrak{p}) + \dim(R/\mathfrak{p}) = \dim(R)$.

(iii): Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $R_{\mathfrak{p}}$ is not a Cohen-Macaulay ring. We claim that $U_R(0)R_{\mathfrak{p}} = U_{R_{\mathfrak{p}}}(0)$. If $(0) = \bigcap_{\mathfrak{q} \in \operatorname{Ass}(R)} I(\mathfrak{q})$ is an irredundant primary decomposition for the zero ideal of R, then

$$(0) = \bigcap_{\substack{\mathfrak{q} \,\in\, \operatorname{Ass}(R) \\ \mathfrak{q} \,\subseteq\, \mathfrak{p}}} I(\mathfrak{q})R_{\mathfrak{p}}$$

is a minimal primary decomposition for the zero ideal of $R_{\mathfrak{p}}$. Thus it is enough to show that

$$\operatorname{Assh}(R_{\mathfrak{p}}) = \big\{\mathfrak{q}R_{\mathfrak{p}} |\ \mathfrak{q} \in \operatorname{Assh}(R),\ \mathfrak{q} \subseteq \mathfrak{p}\big\}.$$

Let $\mathfrak{q} \in \operatorname{Spec}(R)$. We have

$$\begin{aligned}
\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}(R_{\mathfrak{p}}) &\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \dim(R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}}) \\
&\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) = \operatorname{ht}(\mathfrak{p}) \\
&\Longleftrightarrow \mathfrak{q} \subseteq \mathfrak{p} \text{ and } \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim(R/\mathfrak{p}) = \dim(R).
\end{aligned}$$

Let $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}(R_{\mathfrak{p}})$. Since

$$\begin{aligned}
\dim(R/\mathfrak{q}) &\geq \operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim(R/\mathfrak{p}) \\
&= \dim(R),
\end{aligned}$$

$\dim(R/\mathfrak{q}) = \dim(R)$ and hence $\mathfrak{q} \in \operatorname{Assh}(R)$.

Now, let $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \operatorname{Assh}(R)$. Since $R/U_R(0)$ is a catenary ring, $R/\mathfrak{q}$ is also catenary and hence by [8, Theorem 31.4], $\operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim(R/\mathfrak{p}) = \dim(R/\mathfrak{q})$. Thus $\operatorname{ht}(\mathfrak{p}/\mathfrak{q}) + \dim(R/\mathfrak{p}) = \dim(R)$. Therefore, $\mathfrak{q}R_{\mathfrak{p}} \in \operatorname{Assh}(R_{\mathfrak{p}})$.

(iv): Let $\mathfrak{p} \in \operatorname{Spec}(R)$. If $R_{\mathfrak{p}}$ is a Cohen-Macaulay ring, then $R_{\mathfrak{p}}$ is an approximately Cohen-Macaulay ring. If $R_{\mathfrak{p}}$ is not a Cohen-Macaulay ring, then R is not a Cohen-Macaulay ring, and so $R/U_R(0)$ is a Cohen-Macaulay ring. Now, by (iii), $R_{\mathfrak{p}}/U_{R_{\mathfrak{p}}}(0)$ is Cohen-Macaulay. Thus the assertion follows from (i) and Theorem 1(iii). □

By using Theorem 2, the concept of approximately Cohen-Macaulay ring can be extended to nonlocal rings by defining that a ring R is approximately Cohen-Macaulay if for all prime ideals $\mathfrak{p}$ of R, the ring $R_{\mathfrak{p}}$ is an approximately Cohen-Macaulay ring.

In the following result we consider the behavior of $\operatorname{Assh}(-)$ and the primary decomposition of the zero submodule under base change.

Proposition 3 *Let $\varphi: (R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ be a flat local homomorphism. Then the following hold:*

(i) $\operatorname{Assh}(R) = \{\mathfrak{q} \cap R \mid \mathfrak{q} \in \operatorname{Assh}(S)\}$.

(ii) $U_R(0)S \subseteq U_S(0)$.

(iii) *The following conditions are equivalent:*

(a) $U_R(0)S = U_S(0)$.

(b) $\operatorname{Assh}(S) = \{\mathfrak{q} \in \operatorname{Ass}(S) \mid \mathfrak{q} \cap R \in \operatorname{Assh}(R)\}$.

(c) $\operatorname{Ass}_S(S/\mathfrak{p}S) = \operatorname{Assh}_S(S/\mathfrak{p}S)$ *for every* $\mathfrak{p} \in \operatorname{Assh}(R)$.

Proof. (i): Let $\mathfrak{p} \in \operatorname{Assh}(R)$. Then $\dim(R) = \dim(R/\mathfrak{p})$. Consider the flat local homomorphism $\widetilde{\varphi}\ : R/\mathfrak{p} \to S/\mathfrak{p}S$. Hence,

$$\dim(R) = \dim(S/\mathfrak{p}S) - \dim(S/\mathfrak{m}S),$$

and therefore $\dim(S) = \dim(S/\mathfrak{p}S)$. So there exists a minimal prime ideal of $\mathfrak{p}S$, say $\mathfrak{q}$, such that $\dim(S) = \dim(S/\mathfrak{q})$. Since $\mathfrak{q}$ is a minimal prime ideal of $\mathfrak{p}S$, $\mathfrak{q} \cap R = \mathfrak{p}$.

Let $\mathfrak{q} \in \operatorname{Assh}(S)$ and $\mathfrak{q} \cap R = \mathfrak{p}$. We have $\dim(S/\mathfrak{q}) = \dim(S)$ and $\mathfrak{p}S \subseteq \mathfrak{q}$; that means $\dim(S/\mathfrak{p}S) = \dim(S)$. Thus,

$$\dim(R/\mathfrak{p}) + \dim(S/\mathfrak{m}S) = \dim(S),$$

and so $\dim(R/\mathfrak{p}) = \dim(R)$.

(ii): It is enough to show that $U_R(0)S_{\mathfrak{q}} = (0)S_{\mathfrak{q}}$ for every $\mathfrak{q} \in \operatorname{Assh}(S)$. Indeed, since S is flat,

$$\begin{aligned} U_R(0)S_{\mathfrak{q}} &= \bigcap_{\mathfrak{p} \in \operatorname{Assh}(R)} I(\mathfrak{p})S_{\mathfrak{q}} \\ &= \bigcap_{\substack{\mathfrak{p} \,\in\, \operatorname{Assh}(R) \\ \mathfrak{p} \,\subseteq\, \mathfrak{q} \cap R}} I(\mathfrak{p})S_{\mathfrak{q}}. \end{aligned}$$

If $\mathfrak{q} \in \operatorname{Assh}(S)$, then $\mathfrak{q} \cap R = \mathfrak{p}$. We know that $\big(I(\mathfrak{p})R_{\mathfrak{p}}\big)S_{\mathfrak{q}}$ and $(0)S_{\mathfrak{q}}$ are $\mathfrak{p}R_{\mathfrak{p}}$-primary submodules of the $R_{\mathfrak{p}}$-module $S_{\mathfrak{q}}$. Thus $I(\mathfrak{p})S_{\mathfrak{q}} = \big(I(\mathfrak{p})R_{\mathfrak{p}}\big)S_{\mathfrak{q}} = (0)S_{\mathfrak{q}}$, as required.

(iii): It is known that (cf. [8, Theorem 23.2])

$$\{\mathfrak{p}\} = \{\varphi^{-1}(\mathfrak{q}) \mid \mathfrak{q} \in \operatorname{Ass}_S(S/\mathfrak{p}S)\} \text{ for each } \mathfrak{p} \in \operatorname{Spec}(R),$$

$$\mathrm{Ass}(S) = \bigcup_{\mathfrak{p}\in \mathrm{Ass}(R)} \mathrm{Ass}_S(S/\mathfrak{p}S),$$

$$\mathrm{Ass}_S(S/U_R(0)S) = \bigcup_{\mathfrak{p}\in \mathrm{Assh}(R)} \mathrm{Ass}_S(S/\mathfrak{p}S),$$

$$\mathrm{Assh}(S) = \mathrm{Ass}_S(S/U_S(0)) = \bigcup_{\mathfrak{p}\in \mathrm{Assh}(R)} \mathrm{Assh}_S(S/\mathfrak{p}S).$$

Also note that $\cap_{\mathfrak{p}\in \mathrm{Ass}(R)} I(\mathfrak{p})S = (0)$ is a minimal primary decomposition of the zero submodule of the R-module S and if $\cap_{1\leq i\leq n} Q_i = (0)$ is a minimal primary decomposition of the zero ideal in S, then $\cap_{1\leq i\leq n} Q_i = (0)$ is a primary decomposition of the zero submodule of the R-module S. We can now easily obtain (iii). □

The next result shows that the approximately Cohen-Macaulay property is stable under specialization.

Lemma 4 *Let $(R,\mathfrak{m})$ be a Noetherian local ring with $\dim(R) = d$. Let $x \in \mathfrak{m}\backslash Z(R)$. If R is an approximately Cohen-Macaulay ring, then R/xR is an approximately Cohen-Macaulay ring.*

Proof. We may assume that R is not a Cohen-Macaulay ring. Then there exists an ideal I of R such that I is a Cohen-Macaulay R-module, $\dim(I) = d-1$, and R/I is a Cohen-Macaulay ring of dimension d. Therefore, I/xI is a Cohen-Macaulay R/xR-module of dimension $d-2$. Since $\mathrm{Ass}_R(R/I) = \{\mathfrak{q} \in \mathrm{Assh}(R) |\ I \subseteq \mathfrak{q}\}$, $x \notin Z_R(R/I)$ and hence $R/(I+xR)$ is a Cohen-Macaulay ring of dimension $d-1$ and $I\cap xR = xI$. By using the isomorphism $(I+xR)/xR \cong I/(I\cap xR)$, we obtain that $(I+xR)/xR$ is a Cohen-Macaulay R/xR-module of dimension $d-2$. The assertion now follows from Theorem 1. □

Lemma 5 *Let $\varphi :\ (R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ be a flat local homomorphism. Let S be an approximately Cohen-Macaulay ring. Then either R or $S/\mathfrak{m}S$ is Cohen-Macaulay.*

Proof. Assume that R is not Cohen-Macaulay. Then we have

$$\begin{aligned} \mathrm{depth}(R) &= \mathrm{depth}(S) - \mathrm{depth}(S/\mathfrak{m}S) \\ &= \dim(S) - 1 - \mathrm{depth}(S/\mathfrak{m}S) \\ &\geq \dim(S) - 1 - \dim(S/\mathfrak{m}S) \\ &= \dim(R) - 1. \end{aligned}$$

Since R is not Cohen-Macaulay, $\mathrm{depth}(R) = \dim(R) - 1$ and hence $S/\mathfrak{m}S$ is Cohen-Macaulay. □

We are now ready to prove that the approximately Cohen-Macaulay property is stable (in some sense) under change of ring. This result is somehow parallel to the results on properties like regular, complete intersection and Cohen-Macaulay (cf. [10, Theorem 1]).

Theorem 6 *Let* $\varphi:\ (R,\mathfrak{m}) \longrightarrow (S,\mathfrak{n})$ *be a flat local homomorphism. Assume that* R *is not a Cohen-Macaulay ring. Then the following are equivalent:*

(a) *R is an approximately Cohen-Macaulay ring and $S/\mathfrak{m}S$ is a Cohen-Macaulay ring.*

(b) *S is an approximately Cohen-Macaulay ring and* $\operatorname{Ass}_S(S/\mathfrak{p}S) = \operatorname{Assh}_S(S/\mathfrak{p}S)$ *for every* $\mathfrak{p} \in \operatorname{Assh}(R)$.

Further, if R is a homomorphic image of a Cohen-Macaulay local ring, then the next condition is also equivalent:

(c) *S is an approximately Cohen-Macaulay ring.*

Proof. Consider the induced flat local homomorphism $\tilde{\varphi}:\ R/U_R(0) \longrightarrow S/U_R(0)S$.

(a) $\Longrightarrow$ (b): By [3, Theorem 2.1.7], $S/U_R(0)S$ is a Cohen-Macaulay ring. Also the following (in)equalities hold by Proposition 3(ii):

$$\begin{aligned}\dim(S) &\geq \dim\big(S/U_R(0)S\big)\\ &\geq \dim\big(S/U_S(0)\big)\\ &= \dim(S).\end{aligned}$$

On the other hand, $U_R(0)$ is a Cohen-Macaulay R-module of dimension $\dim(R)-1$. Thus $U_R(0)S \cong U_R(0)\otimes_R S$ is a Cohen-Macaulay S-module of dimension $\dim(S)-1$, because

$$\begin{aligned}\dim(S)-1 &= \dim(R)-1+\dim(S/\mathfrak{m}S)\\ &= \dim\big(U_R(0)\big)+\dim(S/\mathfrak{m}S)\\ &= \dim\big(U_R(0)\otimes_R S\big).\end{aligned}$$

The last paragraph of Theorem 1 implies that $U_R(0)S = U_S(0)$. Now, the assertions follow from Proposition 3 and Theorem 1(ii).

(b) $\Longrightarrow$ (a): By Lemma 5, $S/\mathfrak{m}S$ is a Cohen-Macaulay ring. We have $U_S(0) = U_R(0)S$ by Proposition 3(iii). On the other hand, S is not a Cohen-Macaulay ring and so $S/U_R(0)S$ is a Cohen-Macaulay ring. Therefore, $R/U_R(0)$ is a Cohen-Macaulay ring. Now, the assertion follows from the following equalities:

$$\begin{aligned}\dim(R) &= \dim(S)-\dim(S/\mathfrak{m}S)\\ &= \operatorname{depth}(S)+1-\operatorname{depth}(S/\mathfrak{m}S)\\ &= \operatorname{depth}(R)+1.\end{aligned}$$

(b) $\Longrightarrow$ (c): It is clear.

(c) $\Longrightarrow$ (a): By Lemma 5, we know that $S/\mathfrak{m}S$ is Cohen-Macaulay. Assume that $\mathfrak{q}$ is a minimal ideal of $\mathrm{V}(\mathfrak{m}S)$. Then $\mathfrak{q}\cap R=\mathfrak{m}$ and $\dim(S_{\mathfrak{q}}/\mathfrak{m}S_{\mathfrak{q}})=0$. By considering the induced homomorphism $\hat{\varphi}:\ R\longrightarrow S_{\mathfrak{q}}$ one can reduce to the case where $\dim(S/\mathfrak{m}S)=0$. By using "(a) $\Longrightarrow$ (b)" and [5, Corollary 2.6], we may assume that R and S are complete. Note that S is not a Cohen-Macaulay ring. We use induction on $\dim(S)=n$. If $n=1$, then $\dim(R)+\dim(S/\mathfrak{m}S)=1$ and hence $\dim(R)=1$. Thus, R is an approximately Cohen-Macaulay ring. Now suppose, inductively, that $n\geq 2$ and we have established the result for $n-1$. Set $N=\mathrm{Hom}_R\big(\mathrm{H}^{n-1}_{\mathfrak{m}}(R),\mathrm{E}_R(R/\mathfrak{m})\big)$. Since

$$\begin{aligned}\mathrm{depth}(R)\ &=\mathrm{depth}(S)\\ &=\dim(S)-1\\ &=\dim(R)-1,\end{aligned}$$

$\mathrm{H}^{i}_{\mathfrak{m}}(R)=(0)$ for $i\notin\{n-1,n\}$. Therefore, it is enough to show that N is a Cohen-Macaulay R-module of dimension $n-1$.

We claim that $\mathfrak{m}\notin\mathrm{Ass}_R(N)$. Otherwise $\mathfrak{m}\in\mathrm{Att}_R\big(\mathrm{H}^{n-1}_{\mathfrak{m}}(R)\big)$ and so by [2, Exercise 11.3.7], $\mathfrak{n}\in\mathrm{Att}_S\big(\mathrm{H}^{n-1}_{\mathfrak{n}}(S)\big)$. Therefore, $\mathfrak{n}\in\mathrm{Ass}_S\Big(\mathrm{Hom}_S\big(\mathrm{H}^{n-1}_{\mathfrak{n}}(S),\mathrm{E}_S(S/\mathfrak{n})\big)\Big)$. But $\mathrm{Hom}_S\big(\mathrm{H}^{n-1}_{\mathfrak{n}}(S),\mathrm{E}_S(S/\mathfrak{n})\big)$ is a Cohen-Macaulay S-module of dimension $n-1\geq 1$. That is a contradiction.

Since N is a finitely generated R-module, the set $\mathrm{Ass}_R(N)$ is finite and hence there exists $x\in\mathfrak{m}\setminus\big(Z(R)\cup Z_R(N)\big)$. Consider the induced flat local homomorphism $\hat{\varphi}:\ R/xR\longrightarrow S/\varphi(x)S$. Since $\varphi(x)\notin Z(S)$, by Lemma 4, $S/\varphi(x)S$ is an approximately Cohen-Macaulay ring of dimension $n-1$. Therefore, by the inductive hypothesis R/xR is an approximately Cohen-Macaulay module of dimension $n-1$. Set $\bar{R}=R/xR$ and $\bar{\mathfrak{m}}=\mathfrak{m}/xR$. The R/xR-module $\mathrm{Hom}_{\bar{R}}\big(\mathrm{H}^{n-2}_{\bar{\mathfrak{m}}}(\bar{R}),\mathrm{E}_{\bar{R}}(\bar{R}/\bar{\mathfrak{m}})\big)$ is a Cohen-Macaulay module of dimension $n-2$. The exact sequence

$$0\to R\overset{x.}{\to}R\to R/xR\to 0$$

induces the exact sequence

$$0\to\mathrm{H}^{n-2}_{\mathfrak{m}}(R/xR)\to\mathrm{H}^{n-1}_{\mathfrak{m}}(R)\overset{x.}{\to}\mathrm{H}^{n-1}_{\mathfrak{m}}(R).$$

Therefore, $\mathrm{H}^{n-2}_{\bar{\mathfrak{m}}}(\bar{R})\cong\mathrm{Hom}_R\big(R/xR,\mathrm{H}^{n-1}_{\mathfrak{m}}(R)\big)$. By [2, Lemma 10.1.15],

$$\begin{aligned}\mathrm{Hom}_{\bar{R}}\big(\mathrm{H}^{n-2}_{\bar{\mathfrak{m}}}(\bar{R}),\mathrm{E}_{\bar{R}}(\bar{R}/\bar{\mathfrak{m}})\big)\ &\cong\mathrm{Hom}_{\bar{R}}\Big(\mathrm{Hom}_R\big(\bar{R},\mathrm{H}^{n-1}_{\mathfrak{m}}(R)\big),\mathrm{Hom}_R\big(\bar{R},\mathrm{E}_R(R/\mathfrak{m})\big)\Big)\\ &\cong\mathrm{Hom}_R\Big(\mathrm{Hom}_R\big(\bar{R},\mathrm{H}^{n-1}_{\mathfrak{m}}(R)\big),\mathrm{E}_R(R/\mathfrak{m})\Big)\\ &\cong\bar{R}\ \otimes_R\ \mathrm{Hom}_R\big(\mathrm{H}^{n-1}_{\mathfrak{m}}(R),\mathrm{E}_R(R/\mathfrak{m})\big).\end{aligned}$$

Thus N/xN is a Cohen-Macaulay R/xR-module of dimension $n-2$. Since $x \notin Z_R(N)$, N is a Cohen-Macaulay R-module of dimension $n-1$. $\square$

Note that in Theorem 6, the condition "$\mathrm{Ass}_S(S/\mathfrak{p}S) = \mathrm{Assh}_S(S/\mathfrak{p}S)$ for every $\mathfrak{p} \in \mathrm{Assh}(R)$" is not superficial, as the following example shows.

Example 7 (See [9]) Let $(R, \mathfrak{m})$ be a 2-dimensional local domain for which the $\mathfrak{m}$-adic completion $\hat{R} \cong k[[x,y,z]]/(xy, xz)$. Put $S = \hat{R}$. Let $\varphi : R \longrightarrow S$ be a natural ring homomorphism. Then

(i) R is not approximately Cohen-Macaulay local domain. In particular, $\mathrm{Assh}(R) = \{(0)\}$.

(ii) $S/\mathfrak{m}S = k$ is regular, and thus is Cohen-Macaulay.

(iii) S is approximately Cohen-Macaulay, but not unmixed.

(iv) R is not a homomorphic image of a Cohen-Macaulay local ring.

Corollary 8 *Let $\varphi : R \longrightarrow S$ be a flat homomorphism. If R is an approximately Cohen-Macaulay ring and $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_R S$ is a Cohen-Macaulay ring for every $\mathfrak{p} \in \mathrm{Spec}(R)$, then S is an approximately Cohen-Macaulay ring.*

Proof. Let $\mathfrak{q} \in \mathrm{Spec}(S)$. Set $\mathfrak{p} = \mathfrak{q} \cap R \in \mathrm{Spec}(R)$. The induced homomorphism $\tilde{\varphi} : R_\mathfrak{p} \longrightarrow S_\mathfrak{q}$ is a flat local homomorphism. It is clear that $S_\mathfrak{q}/(\mathfrak{p}R_\mathfrak{p})S_\mathfrak{q}$ is a localization of $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_R S$. Now, the assertion follows from Theorem 6. $\square$

Corollary 9 *Let $\varphi : R \longrightarrow S$ is a faithfully flat homomorphism. Suppose that R is not a Cohen-Macaulay ring, but a homomorphic image of a Cohen-Macaulay ring. If S is an approximately Cohen-Macaulay ring, then R is an approximately Cohen-Macaulay ring.*

Proof. Assume that $\mathfrak{p} \in \mathrm{Spec}(R)$ and $\mathfrak{q}$ is a minimal ideal of $\mathrm{V}(\mathfrak{p}S)$. Then $\mathfrak{q} \cap R = \mathfrak{p}$ and $\dim(S_\mathfrak{q}/\mathfrak{p}S_\mathfrak{q}) = 0$. Consider the induced homomorphism $\hat{\varphi} : R_\mathfrak{p} \longrightarrow S_\mathfrak{q}$. The assertion now follows from Theorem 6. $\square$

Theorem 10 *Let R and S be nonzero k-algebras such that $T := R \otimes_k S$ is Noetherian. Assume that R is not a Cohen-Macaulay ring. Then the following hold:*

(i) *If R is an approximately Cohen-Macaulay ring and S is a Cohen-Macaulay ring, then T is an approximately Cohen-Macaulay ring.*

(ii) *If T is an approximately Cohen-Macaulay ring, then S is a Cohen-Macaulay ring.*

(iii) *If R is a homomorphic image of a Cohen-Macaulay ring or k is algebraically closed, then the following conditions are equivalent:*

(a) *T is an approximately Cohen-Macaulay ring.*

(b) *R is an approximately Cohen-Macaulay ring and S is a Cohen-Macaulay ring.*

Proof. (i): Consider the faithfully flat homomorphism $\varphi : R \longrightarrow (R \otimes_k S)$. It is enough to show that the fibers $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_R (R \otimes_k S) \cong (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_k S$ over every prime ideal $\mathfrak{p}$ of R are Cohen-Macaulay rings. Since $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a Cohen-Macaulay ring (it is actually a field), $(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \otimes_k S$ is also a Cohen-Macaulay ring by [10, Theorem 6].

(ii): Assume that S is not a Cohen-Macaulay ring. Then there exist $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathfrak{q} \in \operatorname{Spec}(S)$ such that $R_\mathfrak{p}$ and $S_\mathfrak{q}$ are not Cohen-Macaulay rings, and hence $\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p}) - 1$ and $\operatorname{grade}_{S_\mathfrak{q}}(\mathfrak{q}S_\mathfrak{q}) \leq \operatorname{ht}(\mathfrak{q}) - 1$. Therefore,

$$\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) + \operatorname{grade}_{S_\mathfrak{q}}(\mathfrak{q}S_\mathfrak{q}) \leq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) - 2. \tag{*}$$

There exists $Q \in \operatorname{Spec}(R \otimes_k S)$ such that $Q \cap R = \mathfrak{p}$ and $Q \cap S = \mathfrak{q}$. On the other hand, by [1, Proposition 2.3],

$$\operatorname{ht}(Q) = \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) + \operatorname{ht}\Big(Q/\big((\mathfrak{p} \otimes_k S) + (R \otimes_k \mathfrak{q})\big)\Big)$$

and

$$\begin{aligned}\operatorname{grade}_{(R \otimes_k S)_Q}\big(Q(R \otimes_k S)_Q\big) &= \operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) + \operatorname{grade}_{S_\mathfrak{q}}(\mathfrak{q}S_\mathfrak{q}) \\ &\quad + \operatorname{ht}\Big(Q/\big((\mathfrak{p} \otimes_k S) + (R \otimes_k \mathfrak{q})\big)\Big).\end{aligned}$$

But

$$\operatorname{grade}_{(R \otimes_k S)_Q}\big(Q(R \otimes_k S)_Q\big) \geq \operatorname{ht}(Q) - 1,$$

so we have

$$\operatorname{grade}_{R_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) + \operatorname{grade}_{S_\mathfrak{q}}(\mathfrak{q}S_\mathfrak{q}) \geq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) - 1$$

and by using (*), $\operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) - 1 \leq \operatorname{ht}(\mathfrak{p}) + \operatorname{ht}(\mathfrak{q}) - 2$. That is a contradiction.

(iii): In fact, (b) $\Longrightarrow$ (a) is just (i), and for proving (a) $\Longrightarrow$ (b); by (ii), it is enough to show that R is an approximately Cohen-Macaulay ring.

First, let R be a homomorphic image of a Cohen-Macaulay ring. Consider the faithfully flat homomorphism $\varphi : R \longrightarrow (R \otimes_k S)$. The assertion now follows form Corollary 9.

Next, let k be algebraically closed. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then there exists $\mathfrak{q} \in \operatorname{Spec}(R \otimes_k S)$ with $\mathfrak{q} \cap R = \mathfrak{p}$. Consider the induced flat local homomorphism $\tilde{\varphi} : R_\mathfrak{p} \longrightarrow (R \otimes_k S)_\mathfrak{q}$. Let $\mathfrak{q}' \in \operatorname{Ass}(R \otimes_k S)$ with $\mathfrak{q}' \subseteq \mathfrak{q}$ and $\operatorname{ht}\big(\mathfrak{p}/(\mathfrak{q}' \cap R)\big) = \operatorname{ht}(\mathfrak{p})$. By using Proposition 3 and Theorem 6, it is enough to show that $\operatorname{ht}(\mathfrak{q}/\mathfrak{q}') = \operatorname{ht}(\mathfrak{q})$. Set $\mathfrak{q}' \cap R = \mathfrak{p}'$, $\mathfrak{q} \cap S = \mathfrak{p}_2$, and $\mathfrak{q}' \cap S = \mathfrak{p}_1$. Then by [8, Theorem 23.2], $\mathfrak{q}' \in \operatorname{Ass}_T(T/\mathfrak{p}'T)$. On the other hand, S is Cohen-Macaulay and hence by [10, Theorem

6], $T/\mathfrak{p}'T = (R/\mathfrak{p}') \otimes_k S$ satisfies Serre's condition (S_1). Therefore, $\mathfrak{q}' \in \mathrm{Min}(\mathfrak{p}'T)$ and so $\mathfrak{q}' \in \mathrm{Min}(\mathfrak{p}'T + \mathfrak{p}_1 T)$. On the other hand, $T/(\mathfrak{p}'T + \mathfrak{p}_1 T) \cong (R/\mathfrak{p}') \otimes_k (S/\mathfrak{p}_1)$ is an integral domain (see [4, Exercise A1.2(a), p. 562]). Thus $\mathfrak{q}' = \mathfrak{p}'T + \mathfrak{p}_1 T$. Now, the following equalities hold:

$$\begin{aligned} \mathrm{ht}(\mathfrak{q}/\mathfrak{q}') &= \mathrm{ht}\big(\mathfrak{q}/(\mathfrak{p}'T + \mathfrak{p}_1 T)\big) \\ &= \mathrm{ht}(\mathfrak{p}/\mathfrak{p}') + \mathrm{ht}(\mathfrak{p}_2/\mathfrak{p}_1) + \mathrm{ht}\big(\mathfrak{q}/(\mathfrak{p}T + \mathfrak{p}_2 T)\big) \\ &= \mathrm{ht}(\mathfrak{p}) + \mathrm{ht}(\mathfrak{p}_2) + \mathrm{ht}\big(\mathfrak{q}/(\mathfrak{p}T + \mathfrak{p}_2 T)\big) \\ &= \mathrm{ht}(\mathfrak{q}), \end{aligned}$$

where the second and the last equalities hold by [1, Proposition 2.3] and the third one uses the fact that $\mathfrak{p}_1 \in \mathrm{Ass}(S)$. $\square$

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References

[1] S. Bouchiba, S. E. Kabbaj, *Tensor Products of Cohen-Macaulay Rings: Solution to a Problem of Grothendieck*, J. Algebra **252** (2002), no. 1, 65-73.

[2] M. P. Brodmann, R. Y. Sharp, "*Local Cohomology: An Algebraic Introduction with Geometric Applications*", Cambridge Studies in Advanced Mathematics, 60. Cambridge University Press, Cambridge, 1998.

[3] W. Bruns, J. Herzog, "*Cohen-Macaulay Rings*", Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1993.

[4] D. Eisenbud, "*Commutative Algebra with a View Toward Algebraic Geometry*", Springer-Verlag, 1995.

[5] S. Goto, *Approximately Cohen-Macaulay Rings*, J. Algebra **76** (1982), no. 1, 214-225.

[6] A. Grothendieck, "*Éléments de Géométrie Algébrique*" IV. Étude Locale des Schémas et des Morphismes de Schémas. II. (French) Inst. Hautes Études Sci. Publ. Math., No. 24, 1965.

[7] M. Hochster, *Cyclic Purity Versus Purity in Excellent Noetherian Rings*, Trans. Amer. Math. Soc. **231** (1977), no. 2, 463-488.

[8] H. Matsumura, "*Commutative Ring Theory*", Cambridge Studies in Advanced Mathematics, 8. Cambridge University Press, Cambridge, 1986.

[9] M. Nagata, "*Local Rings*", Interscience Tracts in Pure and Applied Mathematics, No. 13, New York, 1962.

[10] M. Tousi, S. Yassemi, *Tensor Products of Some Special Rings*, J. Algebra **268** (2003), no. 2, 672-676.

[11] K. Watanabe, T. Ishikawa, S. Tachibana, K. Otsuka, *On Tensor Products of Gorenstein Rings*, J. Math. Kyoto Univ. **9** (1969), 413-423.

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