On the Existence of Periodic Solutions for a Class of Generalized Forced Liénard Equations*

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Abstract

In this work the second-order generalized forced Liénard equation $x'' + \big(f(x) + k(x)x'\big)x' + g(x) = p(t)$ is considered and a new condition for guaranteeing the existence of at least one periodic solution for this equation is given.

Keywords: Nonlinear boundary value problem, Liénard equation, Periodic solution, Banach space, Schauder's fixed point theorem.

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1 Introduction

In this work we investigate the existence of periodic solutions for a class of second-order generalized forced Liénard equations

$$x'' + \big(f(x) + k(x)x'\big)x' + g(x) = p(t), \tag{1.1}$$

where f, k, and g are real functions on $\mathbb{R}$ and p is a real function on $[0, T]$, $T > 0$. Generalized forced Liénard equations appear in a number of physical models and an important question is whether these equations can support periodic solutions. This question has been studied extensively by a number of authors; see for example [2, 3, 5, 9, 11, 12, 18, 19, 20]. In particular, there are some existence and multiplicity results for such equations with nonconstant forced terms; see for example [6, 7, 8, 10, 13, 14, 15, 16, 17, 21]. In this direction, we will obtain a new condition to guarantee the existence of at least one periodic solution for (1.1) with a nonconstant forced term. The main purpose of this work is to prove the following result:

Main Theorem *Suppose f, k, and g are real functions on $\mathbb{R}$ which are locally Lipschitz and p is a nonconstant, continuous, real function on $[0, T]$, $T > 0$. Also suppose all solutions of the initial value problem* (1.1) *can be extended to $[0, T]$. If there exist real numbers a_1 and a_2 for which $g(a_1) \leq p(t) \leq g(a_2)$ holds for each $0 \leq t \leq T$, then there exists T_0, $0 < T_0 < T$, such that if p is T_0-periodic, Eq.* (1.1) *has at least one T_0-periodic solution.*

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The rest of the work is organized as follows. In Section 2, we prove that (1.1) has a unique solution satisfying certain conditions by applying Schauder's fixed point theorem. In Section 3, the existence of at least one periodic solution for (1.1) when g has the property mentioned in the Main Theorem is proved.

2 An Existence and Uniqueness Type Result

We start this section by recalling a famous fixed point theorem which was originally due to Schauder: *Let X be a Banach space and Ω be a closed, bounded, and convex subspace of X. If $S : \Omega \to \Omega$ is a compact operator, then S has at least one fixed point on Ω.*

We now state and prove the following existence and uniqueness type result which is a key tool for proving the Main Theorem.

Proposition 2.1 *Let $a_1 < a_2$ and $B > 0$ be real numbers and consider $A = \max\{2|a_1|, 2|a_2|\}$. Suppose f, k, and g are real functions on $\mathbb{R}$ which are locally Lipschitz and at least one of the f, k, or g is nonconstant on $|x| \leq A$; and p is a continuous real function on $[0, T]$, $T > 0$. Also suppose M_0 is the maximum value of $|p|$ on $[0, T]$; M_1, M_2, M_3 are the maximum values of $|f|$, $|k|$, $|g|$ on $|x| \leq A$; and M_1', M_2', M_3' are the Lipschitz constants of f, k, g on $|x| \leq A$, respectively. Consider*

$$M = \tfrac{2}{M_2'B^2+(2M_2+M_1')B+M_3'+M_1},$$

$$N = \tfrac{1}{M_2B^2+M_1B+M_3+M_0}, \text{ and}$$

$$0 < T_0 < \min\left\{T, 2\sqrt{AN}, 2BN, 2\sqrt{M+1}-2\right\}.$$

Then for each $a_1 \leq b \leq a_2$, Eq. (1.1) has a unique solution $x(t)$, satisfying

$$x(0) = x(T_0) = b, \tag{2.1}$$

for which $|x(t)| \leq A$ and $|x'(t)| \leq B$ hold for each $0 \leq t \leq T_0$.

Proof. Consider the equation $x'' = 0$ with boundary condition $x(0) = x(T_0) = b$. The existence of a Green's function for a typical two-endpoint problem was suggested by a simple physical example in [1] and is as follows:

$$G(t, s) = \begin{cases} s(t-T_0)/T_0 & : \text{ if } \quad 0 \leq s \leq t \leq T_0, \\ \\ t(s-T_0)/T_0 & : \text{ if } \quad 0 \leq t \leq s \leq T_0. \end{cases}$$

If we now consider the integral equation

$$x(t) = b + \int_0^{T_0} G(t,s)\Big(\big(f(x(s)) + k(x(s))x'(s)\big)x'(s) + g(x(s)) - p(s)\Big)ds, \quad (2.2)$$

then it is easy to see that the solutions of (2.2) are exactly the solutions of (1.1) satisfying (2.1). Hence, to prove the proposition, it is enough to show that (2.2) has a unique solution $x(t)$ satisfying $|x(t)| \le A$ and $|x'(t)| \le B$ for each $0 \le t \le T_0$. In order to do so, suppose $X = C^1([0,T_0],\mathbb{R})$, and for $\phi \in X$ define

$$||\phi|| = \max_{0\le t\le T_0} |\phi(t)| + \max_{0\le t\le T_0} |\phi'(t)|.$$

It is clear that X is a Banach space. Now, consider

$$\Omega = \Big\{\phi \in X : \ |\phi(t)| \le A \text{ and } \ |\phi'(t)| \le B \text{ hold for each } 0 \le t \le T_0\Big\},$$

which is obviously a closed, bounded, and convex subspace of X. Define the operator $S : \Omega \to X$ by mapping ϕ to $S(\phi)$, where $S(\phi)$ is defined by

$$S(\phi)(t) = b + \int_0^{T_0} G(t,s)\Big(\big(f(\phi(s)) + k(\phi(s))\phi'(s)\big)\phi'(s) + g(\phi(s)) - p(s)\Big)ds.$$

First, we show that S maps Ω into itself. In order to do this, note that for each $x,\ x'$, and t such that $|x| \le A,\ |x'| \le B$, and $0 \le t \le T_0$ we have

$$\begin{aligned} \Big|\big(f(x) + k(x)x'\big)x' + g(x) - p(t)\Big| &\le M_2B^2 + M_1B + M_3 + M_0 \\ &= \tfrac{1}{N}. \end{aligned} \quad (2.3)$$

Also for each $0 \le t \le T_0$ we have

$$\int_0^{T_0} |G(t,s)|ds = \tfrac{1}{2}t(T_0 - t) \le \tfrac{{T_0}^2}{8}, \text{ and}$$

$$\int_0^{T_0} |\tfrac{\partial}{\partial t}G(t,s)|ds = \tfrac{1}{T_0}t^2 - t + \tfrac{1}{2}T_0 \le \tfrac{T_0}{2}.$$

Hence (2.3) implies that for each $\phi \in \Omega$ and $0 \le t \le T_0$,

$$\begin{aligned} |S(\phi)(t)| &\le |b| + \tfrac{1}{N}\int_0^{T_0} |G(t,s)|ds \\ &\le |b| + \tfrac{{T_0}^2}{8N} \\ &\le \tfrac{A}{2} + \tfrac{A}{2} \\ &= A, \text{ and} \end{aligned}$$

$$\begin{aligned} |S(\phi)'(t)| &\le \tfrac{1}{N}\int_0^{T_0} |\tfrac{\partial}{\partial t}G(t,s)|ds \\ &\le \tfrac{T_0}{2N} \\ &\le B. \end{aligned}$$

These mean that for each $\phi \in \Omega$, $S(\phi) \in \Omega$ and therefore S is an operator from Ω to Ω.

Next, we show that S is a compact operator on Ω. For this, it is enough to show that each bounded sequence $\{\phi_n\}$ on Ω has a subsequence $\{\phi_{n_i}\}$ for which $\{S(\phi_{n_i})\}$ is convergent on Ω. Therefore, let $\{\phi_n\}$ be a given sequence on Ω which is automatically bounded by definition of Ω. Suppose $\epsilon > 0$ is given. Since G is a uniformly continuous function on $[0, T_0] \times [0, T_0]$, there exists δ, $0 < \delta < \epsilon N$, such that $(t_1, s_1), (t_2, s_2) \in [0, T_0] \times [0, T_0]$ and $\sqrt{(t_1 - t_2)^2 + (s_1 - s_2)^2} < \delta$ imply that $|G(t_1, s_1) - G(t_2, s_2)| < \epsilon N/2T_0$. By applying (2.3) we now conclude that for each n and for each $t_1, t_2 \in [0, T_0]$, if $|t_1 - t_2| < \delta$, then

$$|S(\phi_n)(t_1) - S(\phi_n)(t_2)| \leq \tfrac{1}{N} \int_0^{T_0} |G(t_1, s) - G(t_2, s)| ds < \epsilon, \text{ and}$$

$$|S(\phi_n)'(t_1) - S(\phi_n)'(t_2)| \leq \tfrac{1}{N} \int_0^{T_0} |\tfrac{\partial}{\partial t} G(t_1, s) - \tfrac{\partial}{\partial t} G(t_2, s)| ds = \tfrac{1}{N} |t_1 - t_2| < \epsilon.$$

Hence $\{S(\phi_n)(t)\}$ and $\{S(\phi_n)'(t)\}$ are equicontinuous families of functions on $[0, T_0]$ and by the classical Ascoli-Arzela theorem, there exists a subsequence $\{\phi_{n_i}(t)\}$ of $\{\phi_n(t)\}$ for which $\{S(\phi_{n_i})(t)\}$ and $\{S(\phi_{n_i})'(t)\}$ are uniformly convergent on $[0, T_0]$. This shows that $\{S(\phi_{n_i})\}$ is convergent on Ω and so S is a compact operator.

Therefore, by Schauder's fixed point theorem, there exists $\phi \in \Omega$ such that $S(\phi) = \phi$. So for each $0 \leq t \leq T_0$, we have $S(\phi)(t) = \phi(t)$ which is to say

$$\phi(t) = b + \int_0^{T_0} G(t, s) \Big(\big(f(\phi(s)) + k((\phi(s)) \phi'(s) \big) \phi'(s) + g(\phi(s)) - p(s) \Big) ds.$$

This means that $\phi \in \Omega$ is a solution of (2.2). Therefore ϕ is a solution of (1.1) which satisfies (2.1) in such a way that $|\phi(t)| \leq A$ and $|\phi'(t)| \leq B$ for each $0 \leq t \leq T_0$.

We now show that ϕ is the unique solution of (1.1) which satisfies the above conditions. Suppose ψ is another solution of (1.1) which satisfies the boundary condition (2.1) such that $|\psi(t)| \leq A$ and $|\psi'(t)| \leq B$ hold for each $0 \leq t \leq T_0$. This means that $\psi \in \Omega$, $\psi \neq \phi$, and $S(\psi) = \psi$. By the locally Lipschitz condition for f, k, and g, note that for each x, y, x', y', and t such that $|x| \leq A$, $|y| \leq A$, $|x'| \leq B$, $|y'| \leq B$, and $0 \leq t \leq T_0$ we have

$$\big| \big((f(x) + k(x)x')x' + g(x) - p(t) \big) - \big((f(y) + k(y)y')y' + g(y) - p(t) \big) \big| =$$

$$\big| \big(f(x) - f(y) \big) x' + f(y)(x' - y') + \big(k(x) - k(y) \big) x'^2 + \ k(y)(x'^2 - y'^2) + g(x) - g(y) \big| \leq$$

$$(M_2' B^2 + M_1' B + M_3') |x - y| + (2M_2 B + M_1) |x' - y'|.$$

Therefore by the above inequality, for each $0 \le t \le T_0$,

$$
\begin{aligned}
|S(\phi)(t) - S(\psi)(t)| &\le \tfrac{T_0{}^2}{8}\big(M_2' B^2 + (2M_2 + M_1')B + M_3' + M_1\big)||\phi - \psi|| \\
&= \tfrac{T_0{}^2}{8}\tfrac{2}{M}||\phi - \psi|| \\
&= \tfrac{T_0{}^2}{4M}||\phi - \psi||, \text{ and} \\
|S(\phi)'(t) - S(\psi)'(t)| &\le \tfrac{T_0}{2}\big(M_2' B^2 + (2M_2 + M_1')B + M_3' + M_1\big)||\phi - \psi|| \\
&= \tfrac{T_0}{2}\tfrac{2}{M}||\phi - \psi|| \\
&= \tfrac{T_0}{M}||\phi - \psi||.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
||\phi - \psi|| &= ||S(\phi) - S(\psi)|| \\
&= \max_{0 \le t \le T_0} |S(\phi)(t) - S(\psi)(t)| + \max_{0 \le t \le T_0} |S(\phi)'(t) - S(\psi)'(t)| \\
&\le (\tfrac{T_0{}^2}{4M} + \tfrac{T_0}{M})||\phi - \psi||.
\end{aligned}
$$

Therefore we obtain $T_0{}^2 + 4T_0 \ge 4M$, or $T_0 \ge 2\sqrt{M+1} - 2$ which is contradictory with the definition of T_0. So ϕ is the unique solution of (1.1), satisfying the given conditions. $\square$

The above proposition implies the following existence result.

Corollary 2.2 *Let k be a locally Lipschitz real function on $\mathbb{R}$ which is nonconstant on each compact interval. Then for each given $T_0 > 0$ and b, the following boundary value problem:*

$$
\begin{cases}
x'' + k(x)x'^2 = 0, \\
x(0) = x(T_0) = b,
\end{cases}
$$

has a solution.

Proof. We apply Proposition 2.1 with $p = 0$, say defined on $[0, T]$, $T > 0$. Suppose a_1 and a_2 are two real numbers such that $a_1 < b < a_2$ and consider $A = \max\{2|a_1|, 2|a_2|\}$. Let $B > 0$ be arbitrary. Suppose M_2 is the maximum value of $|k|$ on $|x| \le A$ and M_2' is the Lipschitz constant of k on $|x| \le A$. Consider

$$
M = \tfrac{2}{M_2' B^2 + 2M_2 B},
$$

$$
N = \tfrac{1}{M_2 B^2},
$$

and choose B small enough and also T large enough such that

$$T_0 < \min\left\{T,\ \frac{2\sqrt{A}}{B\sqrt{M_2}},\ \frac{2}{M_2 B},\ 2\sqrt{\frac{2}{M_2' B^2 + 2M_2 B} + 1} - 2\right\}.$$

Proposition 2.1 now implies that the given boundary value problem has a solution. Note that this solution with restrictions $|x(t)| \leq A$ and $|x'(t)| \leq B$ for each $0 \leq t \leq T_0$ is unique. $\square$

3 Proof of the Main Theorem

In this section we prove the Main Theorem. By the assumption we conclude $a_1 \neq a_2$ and so without loss of generality we can suppose that $a_1 < a_2$. Define the functions $\tilde{g}$ and $\hat{g}$, which are obviously locally Lipschitz, as follows:

$$\tilde{g}(x) = \begin{cases} g(x) & : \text{ if } x \leq a_1, \\ g(a_1) + a_1 - x & : \text{ if } x > a_1, \end{cases}$$

and

$$\hat{g}(x) = \begin{cases} g(x) & : \text{ if } x \geq a_2, \\ g(a_2) + a_2 - x & : \text{ if } x < a_2. \end{cases}$$

Consider $A = \max\{2|a_1|, 2|a_2|\}$ and suppose $B > 0$ is arbitrary. Let M_0 be the maximum value of $|p|$ on $[0, T]$; $M_1,\ M_2,\ M_3,\ \tilde{M}_3,\ \hat{M}_3$ be the maximum values of $|f|,\ |k|,\ |g|,\ |\tilde{g}|,\ |\hat{g}|$ on $|x| \leq A$; and $M_1',\ M_2',\ M_3',\ \tilde{M}_3',\ \hat{M}_3'$ be the Lipschitz constants of $f,\ k,\ g,\ \tilde{g},\ \hat{g}$ on $|x| \leq A$, respectively. Consider

$$\begin{aligned}
&M = \tfrac{2}{M_2' B^2 + (2M_2 + M_1')B + M_3' + M_1}, \\
&N = \tfrac{1}{M_2 B^2 + M_1 B + M_3 + M_0}, \\
&\tilde{M} = \tfrac{2}{M_2' B^2 + (2M_2 + M_1')B + \tilde{M}_3' + M_1}, \\
&\tilde{N} = \tfrac{1}{M_2 B^2 + M_1 B + \tilde{M}_3 + M_0}, \\
&\hat{M} = \tfrac{2}{M_2' B^2 + (2M_2 + M_1')B + \hat{M}_3' + M_1}, \\
&\hat{N} = \tfrac{1}{M_2 B^2 + M_1 B + \hat{M}_3 + M_0}, \text{ and} \\
&0 < T_0 < \min\{L, \tilde{L}, \hat{L}\}, \text{ where} \\
&L = \min\left\{T, 2\sqrt{AN}, 2BN, 2\sqrt{M+1} - 2\right\}, \\
&\tilde{L} = \min\left\{T, 2\sqrt{A\tilde{N}}, 2B\tilde{N}, 2\sqrt{\tilde{M}+1} - 2\right\}, \text{ and} \\
&\hat{L} = \min\left\{T, 2\sqrt{A\hat{N}}, 2B\hat{N}, 2\sqrt{\hat{M}+1} - 2\right\}.
\end{aligned}$$

Proposition 2.1 now implies that for each $a_1 \le b \le a_2$, the Eq. (1.1) has a unique solution, say $x_b(t)$, satisfying $x_b(0) = x_b(T_0) = b$ for which $|x_b(t)| \le A$ and $|x_b'(t)| \le B$ hold for each $0 \le t \le T_0$.

Lemma 3.1 *For each $0 \le t \le T_0$, we have $x_{a_1}(t) \le a_1 < a_2 \le x_{a_2}(t)$.*

Proof. First, we prove that $x_{a_1}(t) \le a_1$ holds for each $0 \le t \le T_0$. By Proposition 2.1, the equation

$$x'' + \big(f(x) + k(x)x'\big)x' + \tilde{g}(x) = p(t)$$

has a unique solution $x(t)$ satisfying $x(0) = x(T_0) = a_1$ for which $|x(t)| \le A$ and $|x'(t)| \le B$ hold for each $0 \le t \le T_0$. We claim that $x(t) \le a_1$ holds for each $0 \le t \le T_0$. Suppose, for the purpose of a contradiction, there exists a point $0 \le \tilde{t} \le T_0$ such that $x(\tilde{t}) > a_1$. Therefore the function $x(t) - a_1$ has a positive maximum on the interval $(0, T_0)$, say at t_1. Hence $(x(t) - a_1)'\big|_{t=t_1} = 0$, or $x'(t_1) = 0$. Therefore we have established

$$\begin{aligned}
x''(t_1) &= -\big(f(x(t_1)) + k(x(t_1))x'(t_1)\big)x'(t_1) - \tilde{g}(x(t_1)) + p(t_1)\\
&= -\tilde{g}(x(t_1)) + p(t_1)\\
&= -g(a_1) - a_1 + x(t_1) + p(t_1)\\
&= \big(p(t_1) - g(a_1)\big) + \big(x(t_1) - a_1\big)\\
&> 0.
\end{aligned}$$

This implies that $(x(t) - a_1)''\big|_{t=t_1} > 0$, which is a contradiction since $x(t) - a_1$ has a maximum at t_1. Therefore for each $0 \le t \le T_0$, $x(t) \le a_1$ and so by the definition of $\tilde{g}$, $\tilde{g}(x(t)) = g(x(t))$ holds for each $0 \le t \le T_0$. This means that $x(t)$ is a solution of (1.1) satisfying $x(0) = x(T_0) = a_1$ for which $|x(t)| \le A$ and $|x'(t)| \le B$ hold for each $0 \le t \le T_0$. The uniqueness property now implies that for each $0 \le t \le T_0$, $x(t) = x_{a_1}(t)$ and so $x_{a_1}(t) \le a_1$ holds for each $0 \le t \le T_0$.

Next, we prove that $a_2 \le x_{a_2}(t)$ holds for each $0 \le t \le T_0$. By Proposition 2.1, the equation

$$x'' + \big(f(x) + k(x)x'\big)x' + \hat{g}(x) = p(t)$$

has a unique solution $x(t)$ satisfying $x(0) = x(T_0) = a_2$ for which $|x(t)| \le A$ and $|x'(t)| \le B$ hold for each $0 \le t \le T_0$. We claim that $a_2 \le x(t)$ holds for each $0 \le t \le T_0$. Suppose, for the purpose of a contradiction, there exists a point $0 \le \hat{t} \le T_0$ such that $a_2 > x(\hat{t})$. Therefore the function $x(t) - a_2$ has a negative minimum on the interval $(0, T_0)$, say at t_2. Hence $(x(t) - a_2)'\big|_{t=t_2} = 0$, or $x'(t_2) = 0$. Therefore we have established

$$
\begin{aligned}
x''(t_2) &= -\big(f(x(t_2)) + k(x(t_2))x'(t_2)\big)x'(t_2) - \hat{g}(x(t_2)) + p(t_2) \\
&= -\hat{g}(x(t_2)) + p(t_2) \\
&= -g(a_2) - a_2 + x(t_2) + p(t_2) \\
&= \big(p(t_2) - g(a_2)\big) + \big(x(t_2) - a_2\big) \\
&< 0.
\end{aligned}
$$

This implies that $(x(t) - a_2)''\big|_{t=t_2} < 0$, which is a contradiction since $x(t) - a_2$ has a minimum at t_2. Therefore for each $0 \le t \le T_0$, $a_2 \le x(t)$ and so by the definition of $\hat{g}$, $\hat{g}(x(t)) = g(x(t))$ holds for each $0 \le t \le T_0$. This means that $x(t)$ is a solution of (1.1) satisfying $x(0) = x(T_0) = a_2$ for which $|x(t)| \le A$ and $|x'(t)| \le B$ hold for each $0 \le t \le T_0$. The uniqueness property now implies that for each $0 \le t \le T_0$, $x(t) = x_{a_2}(t)$ and so $a_2 \le x_{a_2}(t)$ holds for each $0 \le t \le T_0$. $\square$

Lemma 3.2 *There exists $\hat{b}$, $a_1 \le \hat{b} \le a_2$, such that ${x'}_{\hat{b}}(0) = {x'}_{\hat{b}}(T_0)$.*

Proof. Define the function θ on $[a_1, a_2]$ by

$$\theta(b) = {x'}_b(0) - {x'}_b(T_0).$$

Using the Ascoli-Arzela theorem, one may easily verify that both $x_b(t)$ and ${x'}_b(t)$ are continuous on $[0, T_0] \times [a_1, a_2]$. This implies that θ is continuous also. On the other hand, note that for $i \in \{1, 2\}$,

$$x'_{a_i}(0) = \lim_{t \to 0^+} \frac{x_{a_i}(t) - a_i}{t}, \qquad x'_{a_i}(T_0) = \lim_{t \to 0^+} \frac{a_i - x_{a_i}(T_0 - t)}{t},$$

and therefore,

$$
\begin{aligned}
\theta(a_i) &= {x'}_{a_i}(0) - {x'}_{a_i}(T_0) \\
&= \lim_{t \to 0^+} \frac{x_{a_i}(t) + x_{a_i}(T_0 - t) - 2a_i}{t}.
\end{aligned}
$$

So by Lemma 3.1, we obtain $\theta(a_1) \le 0$ and $\theta(a_2) \ge 0$. Hence there exists $\hat{b}$, $a_1 \le \hat{b} \le a_2$, such that $\theta(\hat{b}) = 0$, or ${x'}_{\hat{b}}(0) = {x'}_{\hat{b}}(T_0)$. $\square$

Therefore $x_{\hat{b}}(t)$ is a solution of (1.1) satisfying the following periodic boundary conditions:

$$
\begin{aligned}
x_{\hat{b}}(0) &= x_{\hat{b}}(T_0), \\
{x'}_{\hat{b}}(0) &= {x'}_{\hat{b}}(T_0).
\end{aligned}
$$

By a method similar to the one used in [4], we now extend $x_{\hat{b}}(t)$ periodically with period T_0 to obtain a periodic solution of the Eq. (1.1). Note that this periodic solution is nontrivial, since p is a nonconstant forced function. $\square$

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References

[1] G. Birkhoff, G. G. Rota, "*Ordinary Differential Equations*", Third ed., John Wiley and Sons, 1978.

[2] A. Capietto, Z. Wang, *Periodic Solutions of Liénard Equations with Asymmetric Nonlinearities at Resonance*, J. London Math. Soc. (2) **68** (2003), no. 1, 119-132.

[3] A. Capietto, Z. Wang, *Periodic Solutions of Liénard Equations at Resonance*, Differential Integral Equations **16** (2003), no. 5, 605-624.

[4] L. Cesari, "*Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*", Second ed., Academic Press, New York, 1963.

[5] S. K. Chang, H. S. Chang, *Existence of Periodic Solutions of Nonlinear Systems of a Generalized Liénard Type*, Kyungpook Math. J. **39** (1999), no. 2, 351-365.

[6] H. B. Chen, K. T. Li, D. S. Li, *Existence of Exactly One and Two Periodic Solutions of the Liénard Equation*, Acta Math. Sinica **47** (2004), no. 3, 417-424.

[7] H. B. Chen, Y. Li, *Exact Multiplicity for Periodic Solutions of a First-Order Differential Equation*, J. Math. Anal. Appl. **292** (2004), no. 2, 415-422.

[8] H. B. Chen, Y. Li, X. Hou, *Exact Multiplicity for Periodic Solutions of Duffing Type*, Nonlinear Anal. **55** (2003), no. 1-2, 115-124.

[9] E. Esmailzadeh, B. Mehri, G. Nakhaie-Jazar, *Periodic Solution of a Second Order, Autonomous, Nonlinear System*, Nonlinear Dynam. **10** (1996), no. 4, 307-316.

[10] C. Fabry, J. Mawhin, M. N. Nkashama, *A Multiplicity Result for Periodic Solutions of Forced Nonlinear Second Order Ordinary Differential Equations*, Bull. London Math. Soc. **18** (1986), no. 2, 173-180.

[11] J. F. Jiang, *On the Qualitative Behavior of Solutions of the Equation* $\ddot{x}+f_1(x)\dot{x}+f_2(x)\dot{x}^2+g(x)=0$, J. Math. Anal. Appl. **194** (1995), no. 3, 597-611.

[12] J. F. Jiang, *The Global Stability of a Class of Second Order Differential Equations*, Nonlinear Anal. **28** (1997), no. 5, 855-870.

[13] G. Katriel, *Uniqueness of Periodic Solutions for Asymptotically Linear Duffing Equations with Strong Forcing*, Topol. Methods Nonlinear Anal. **12** (1998), no. 2, 263-274.

[14] A. C. Lazer, P. J. McKenna, *On the Existence of Stable Periodic Solutions of Differential Equations of Duffing Type*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 125-133.

[15] J. Mawhin, "*Topological Degree and Boundary Value Problems for Nonlinear Differential Equations*", Lecture Notes in Mathematics, vol. 1537, Springer-Verlag, Berlin, 1993.

[16] R. Ortega, *Stability and Index of Periodic Solutions of an Equation of Duffing Type*, Boll. Un. Mat. Ital. B (7) **3** (1989), no. 3, 533-546.

[17] G. Tarantello, *On the Number of Solutions for the Forced Pendulum Equation*, J. Differential Equations **80** (1989), no. 1, 79-93.

[18] G. Villari, *On the Existence of Periodic Solutions for Liénard's Equation*, Nonlinear Anal. **7** (1983), no. 1, 71-78.

[19] Z. H. Zheng, *Periodic Solutions of Generalized Liénard Equations*, J. Math. Anal. Appl. **148** (1990), no. 1, 1-10.

[20] J. Zhou, *On the Existence and Uniqueness of Periodic Solutions for Liénard-Type Equations*, Nonlinear Anal. **27** (1996), no. 12, 1463-1470.

[21] A. Zitan, R. Ortega, *Existence of Asymptotically Stable Periodic Solutions of a Forced Equation of Liénard Type*, Nonlinear Anal. **22** (1994), no. 8, 993-1003.

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