PROBABILITY THAT THE COMMUTATOR OF TWO GROUP ELEMENTS IS EQUAL TO A GIVEN ELEMENT

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Abstract. In this paper we study the probability that the commutator of two randomly chosen elements in a finite group is equal to a given element of that group. Explicit computations are obtained for groups G which $|G'|$ is prime and $G' \leq Z(G)$ as well as for groups G which $|G'|$ is prime and $G' \cap Z(G) = 1$. This paper extends results of Rusin [see D. J. Rusin, What is the probability that two elements of a finite group commute? Pacific J. Math. 82 (1979), no. 1, 237–247].

1. Introduction

Let G be a finite group acting on the finite set Ω. Assume that $\mathrm{Fix}(G, \Omega)$ is the set of pairs (g, ω) of $G \times \Omega$ such that $g.\omega = \omega$ and put $\mathrm{p}_G(\Omega) = |\mathrm{Fix}(G, \Omega)|/|G||\Omega|$. Of course, this quantity is the probability that an element of G leaves an element of Ω fixed. This concept has been considered by several authors. Sherman [13] considered the case for which G is abelian and A its full automorphism group. Therefore, $\mathrm{p}_A(G)$ is the probability that an automorphism leaves an element fixed. He proved for groups of order p^n, p prime, $\mathrm{p}_A(G) \leq 2(3/p^2)^{n/2}$. He further showed that if G_n is a sequence of abelian groups with $|G_n| \to \infty$ then $\mathrm{p}_A(G_n) \to 0$.

If we assume G acts on itself by conjugation, then one obtains $\mathrm{p}_G(G)$, simply denoted by $\mathrm{p}(G)$, is the probability that two randomly chosen elements of G commute. Note that several other notations have been used for $\mathrm{p}(G)$ such as $\mathrm{Pr}(G)$, $\mathrm{R}(G)$, $\mathrm{mc}(G)$, and etc. An important formula for this latter probability is $\mathrm{p}(G) = k(G)/|G|$, where $k(G)$ is the number of conjugacy classes of G. The proof of this equality has been established by Gustafson [6] and the technique of the proof was used by Erdös and Turán [3]. He showed that the maximum value for $k(G)/|G|$ for finite nonabelian groups is $5/8$, so $\mathrm{p}(G) \leq 5/8$ for all such groups which is, of course, sharp. Also he showed that, with a suitable interpretation, this latter inequality remains valid for compact topological nonabelian groups. Rusin [12] again considered $\mathrm{p}(G) = k(G)/|G|$, the probability that two elements of G commute, and has been obtained an explicit computation of $\mathrm{p}(G)$ for groups G with $G' \leq Z(G)$. Also some limiting conditions were given and he classified the groups G for which $\mathrm{p}(G)$ is greater than $11/32$.

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Now let G be a finite group and g be an element of G and consider the probability $\mathrm{p}_g(G)$ that the commutator of two randomly chosen elements of G is equal to g. It is clear that $\mathrm{p}_1(G)$ is equal to the probability that two randomly chosen elements of G commute and, therefore, this latter probability is a generalization of $\mathrm{p}(G)$. In this paper we study $\mathrm{p}_g(G)$. There are two possibilities for groups G in which $|G'|$ is prime: $G' \leq Z(G)$ or $G' \cap Z(G) = 1$. Explicit computations for $\mathrm{p}_g(G)$ are obtained in each cases (see Sections 3 and 4). This paper extends results of Rusin [12].

2. Some computing formulas

In this section we give some formulas to compute the probability $\mathrm{p}_g(G)$ that the commutator of two randomly chosen elements in a finite group G is equal to a given element g of G. Obviously for those g's lie in $G \setminus G'$, we have $\mathrm{p}_g(G) = 0$. Therefore in the sequel we deal only with g's which lie in G'. Note that there are examples of groups G of order 96 due to Guralnick [5] where $\mathrm{p}_g(G) = 0$ even when g belongs to G'.

We now start with the following theorem which gives us a character theoretical formula for $\mathrm{p}_g(G)$. In the following $\mathrm{Irr}(G)$ denotes the set of all irreducible complex characters of G.

Theorem 2.1. *Let G be a finite group, and let $g \in G'$. Then we have*

$$\mathrm{p}_g(G) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)}.$$

Proof. Let D be the set of pairs (x, y) of $G \times G$ such that $[x, y] = g$, where $[x, y] := x^{-1}y^{-1}xy$ denotes the commutator of x, y. Therefore we have $\mathrm{p}_g(G) = |D|/|G|^2$. On the other hand, by a theorem of Frobenius [4] the number of solutions of the equation $[x, y] = g$ in G, i.e., $|D|$, is equal to $|G| \sum_{\chi \in \mathrm{Irr}(G)} \chi(g)/\chi(1)$. Therefore we obtain

$$\mathrm{p}_g(G) = \frac{|D|}{|G|^2} = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)},$$

as required. □

Let us note that if we consider $g = 1$ in the above theorem, then we obtain

$$\mathrm{p}_1(G) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(1)}{\chi(1)} = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} 1 = \frac{|\mathrm{Irr}(G)|}{|G|} = \frac{k(G)}{|G|},$$

where $k(G)$ is the number of conjugacy classes of G. The quantity $\mathrm{p}_1(G)$ is the probability that two randomly chosen elements of G commute and so the latter equality is a well known result, as we mentioned in the previous section. Therefore Theorem 2.1 is a generalization of this well known key result.

Given a finite group G, let $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$ be the set of degrees of the irreducible complex characters of G. The study of the structure of a finite group G by imposing conditions on the set $\mathrm{cd}(G)$ has been considered in many papers in the last decades. For example, groups having just two different irreducible complex character degrees are solvable, and these groups have been thoroughly investigated (see the results by Isaacs and Passman in [10, 11], Isaacs' book [9, Chapter 12] or Bannuscher's papers [1, 2]). In the following theorem we give a formula to compute $\mathrm{p}_g(G)$ for finite groups with just two irreducible complex character degrees.

Theorem 2.2. *Let G be a finite group such that $\mathrm{cd}(G) = \{1, m\}$, $m > 1$. If $g \in G'$, then we have*

$$\mathrm{p}_g(G) = \begin{cases} \dfrac{1}{|G'|}\left(1 - \dfrac{1}{m^2}\right) & \textit{if } g \neq 1, \\ \\ \dfrac{1}{|G'|}\left(1 + \dfrac{|G'| - 1}{m^2}\right) & \textit{if } g = 1. \end{cases}$$

Proof. First, we assume that $g \neq 1$. By applying second orthogonality relation for g we obtain that

$$0 = \sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\chi(1) = \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) = 1}} \chi(g)\chi(1) + \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) > 1}} \chi(g)\chi(1).$$

In the case that χ is linear, we have $G' \leq \mathrm{Ker}\ \chi$ and therefore $g \in \mathrm{Ker}\ \chi$. Hence $\chi(g) = \chi(1)$. Also the number of all linear irreducible complex characters of G is equal to $|G : G'|$. On the other hand, by assumption, $\chi(1) = m$ holds for each nonlinear irreducible complex character χ of G. Therefore we have

$$\sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) = 1}} \chi(g)\chi(1) = \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) = 1}} \chi(1)^2 = \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) = 1}} 1 = |G : G'|,$$

which implies that

$$|G : G'| + m \sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) > 1}} \chi(g) = 0.$$

Therefore we get

$$\sum_{\substack{\chi \in \mathrm{Irr}(G) \\ \chi(1) > 1}} \chi(g) = -\frac{|G : G'|}{m},$$

and so by Theorem 2.1 we have

$$\begin{aligned}
\mathrm{p}_g(G) &= \frac{1}{|G|}\sum_{\chi\in\mathrm{Irr}(G)}\frac{\chi(g)}{\chi(1)} \\
&= \frac{1}{|G|}\Bigg(\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi(1)=1}}\frac{\chi(g)}{\chi(1)}+\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi(1)>1}}\frac{\chi(g)}{\chi(1)}\Bigg) \\
&= \frac{1}{|G|}\Bigg(\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi(1)=1}}1+\frac{1}{m}\sum_{\substack{\chi\in\mathrm{Irr}(G)\\ \chi(1)>1}}\chi(g)\Bigg) \\
&= \frac{1}{|G|}\Bigg(|G:G'|+\frac{1}{m}\Big(-\frac{|G:G'|}{m}\Big)\Bigg) \\
&= \frac{1}{|G'|}\Big(1-\frac{1}{m^2}\Big),
\end{aligned}$$

as required. Second, we assume that $g=1$. In this case, $\mathrm{p}_1(G)$ is the probability that two randomly chosen elements of G commute, i.e., it is equal to $k(G)/|G|$, where $k(G)$ is the number of conjugacy classes of G. We now have

$$|G| = \sum_{\chi\in\mathrm{Irr}(G)}\chi(1)^2 = |G:G'| + \Big(k(G)-|G:G'|\Big)m^2,$$

so we obtain

$$\mathrm{p}_1(G) = \frac{1}{|G'|}\Big(1+\frac{|G'|-1}{m^2}\Big)$$

as required. □

It is well known that the degree of an irreducible complex character cannot exceed $|G:Z(G)|^{1/2}$. Finite groups for which this bound is attained are called groups of *central type*. In 1982 Howlett and Isaacs [8] proved that a finite group of central type must be solvable, but not necessarily nilpotent. Now in view of Theorem 2.2, we can state the following corollary.

Corollary 2.3. *Let G be a finite group such that $|\mathrm{cd}(G)|=2$. If $g\in G'$ is nonidentity, then we have*

$$\mathrm{p}_g(G) \le \frac{1}{|G'|}\Big(1-\frac{1}{|G:Z(G)|}\Big).$$

Also we have

$$\mathrm{p}_1(G) \ge \frac{1}{|G'|}\Big(1+\frac{|G'|-1}{|G:Z(G)|}\Big).$$

In both cases the equalities hold if and only if G is a group of central type.

Finite groups with just two irreducible complex character degrees are nonabelian, so for such groups the index of centre is greater than or equal to four. Therefore we can obtain the following corollary which gives us bounds on $\mathrm{p}_g(G)$ for a class of such groups.

Corollary 2.4. *Let G be a finite group of central type such that $|\mathrm{cd}(G)| = 2$. If $g \in G'$ is nonidentity, then we have $\mathrm{p}_g(G) \geq 3/4|G'|$. Also we have $\mathrm{p}_1(G) \leq (|G'|+3)/4|G'|$.*

3. Groups G which $|G'|$ is prime and $G' \leq Z(G)$

In this section we deal with finite groups G which $|G'|$ is prime and $G' \leq Z(G)$. Note that these groups are finite nilpotent groups with nilpotency class at most two whose derived subgroups are cyclic of prime order. Using Theorem 2.2, we can obtain the following proposition which gives us an explicit formula to compute $\mathrm{p}_g(G)$ for such groups.

Proposition 3.1. *Let G be a finite group such that $|G'| = p$, p prime, and let $G' \leq Z(G)$. If $g \in G'$, then we have*

$$\mathrm{p}_g(G) = \begin{cases} \dfrac{1}{p}\left(1 - \dfrac{1}{|G:Z(G)|}\right) & \text{if } g \neq 1, \\[2ex] \dfrac{1}{p}\left(1 + \dfrac{p-1}{|G:Z(G)|}\right) & \text{if } g = 1. \end{cases}$$

Moreover, if $g \in G'$ is nonidentity, then we have $\mathrm{p}_g(G) \geq 3/4p$. Also we have $\mathrm{p}_1(G) \leq (p+3)/4p$.

Proof. Let χ be a nonlinear irreducible complex character of G. We claim that χ vanishes outside of $Z(G)$. In order to prove the claim suppose that $g \in G \setminus Z(G)$. Therefore there exists an element $x \in G$ such that $z := [g,x] \neq 1$. Since $|G'| = p$ is a prime, each nontrivial element of G' is a generator for G' and so $G' = \langle z \rangle$. Now consider ρ as a $\mathbb{C}$-representation of G affording χ. By [9, Lemma 2.27] we have $\rho(z) = \varepsilon I$ for some $\varepsilon \in \mathbb{C}$. In the case $\varepsilon = 1$, we have $z \in \mathrm{Ker}\ \rho$ and therefore $G' \leq \mathrm{Ker}\ \rho$ which is a contradiction by [9, Corollary 2.23] and hypothesis. Therefore $\varepsilon \neq 1$ and since $\chi(g) = \chi(x^{-1}gx) = \chi(gz) = \mathrm{tr}\ \rho(gz) = \mathrm{tr}(\rho(g)\rho(z)) = \mathrm{tr}(\varepsilon\rho(g)I) = \varepsilon\chi(g)$, we obtain $\chi(g) = 0$, so the claim holds. Corollary 2.30 of [9] now implies that $\chi(1)^2 = |G : Z(G)|$. Therefore G is a group of central type with just two irreducible complex character degrees, i.e., $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. Now the assertion holds by Theorem 2.2 and Corollary 2.4. □

For developing our results we need the notion of isoclinism, for which we refer the reader to [7]. Note that isoclinism is an equivalence relation between groups. It is obvious that if G and H are isomorphic groups, then they are isoclinic. In what follows we state an important result which is proved in [7], that will be useful for further considerations.

Proposition 3.2. *Let G be a group. Then there is a group H isoclinic to G such that $Z(H) \leq H'$. If G is finite, so is any such H.*

For a given finite group G, the above proposition implies that the set $\mathbb{ISO}(G)$ which is defined by $\{H : H \text{ is a finite group isoclinic to } G \text{ such that } Z(H) \leq H'\}$ is nonempty. Let us state the following lemma which will be useful for further considerations.

Lemma 3.3. *Let G be a finite group such that $G' \leq Z(G)$. Then we have $\mathbb{ISO}(G) = \{H : H \text{ is a finite group isoclinic to } G \text{ such that } H' = Z(H)\}$. Moreover, if we assume that $|G'| = p$, p prime, then $\{|H| : H \in \mathbb{ISO}(G)\} = \{p^n\}$ for some positive integer n.*

Proof. If $H \in \mathbb{ISO}(G)$, then G and H are isoclinic, say via isoclinism (φ, ψ), and we have $Z(H) \leq H'$. By assumption, $G' \leq Z(G)$, therefore $G/Z(G)$ is abelian and since $G/Z(G) \cong H/Z(H)$ (via φ), we obtain that $H/Z(H)$ is abelian which implies that $H' \leq Z(H)$. Hence we have $H' = Z(H)$. This proves the first part of the lemma. For the proof of the second part of the lemma suppose that H and K be elements of $\mathbb{ISO}(G)$. We have then $|H/H'| = |H/Z(H)| = |K/Z(K)| = |K/K'|$. But $|H'| = |K'|$, so we have $|H| = |K|$ which implies that $\{|H| : H \in \mathbb{ISO}(G)\} = \{l\}$ for some positive integer l. Now suppose that $|G'| = p$ and H be an element of $\mathbb{ISO}(G)$. If q $(\neq p)$ is a prime divisor of $|H|$, nilpotency of H implies that the Sylow q-subgroup of H must be lie in $Z(H) = H'$ which is a contradiction since H' is a p-group. Therefore H must be a p-group and so l has the form p^n for some positive integer n which completes the proof of the lemma. □

Viewing the above lemma, for the finite group G such that $|G'| = p$, p prime, and $G' \leq Z(G)$, the positive integer n for which $\{|H| : H \in \mathbb{ISO}(G)\} = \{p^n\}$ is called *isoclinic exponent* of G and denoted by $\mathrm{iso.exp}(G)$.

We can now state the following proposition for a finite group G, in terms of isoclinic exponent of G, which gives us a formula for $\mathrm{p}_g(G)$ analogue to Proposition 3.1.

Proposition 3.4. *Let G be a finite group such that $|G'| = p$, p prime, and let $G' \leq Z(G)$. Suppose that* $\mathrm{iso.exp}(G) = n$. *If $g \in G'$, then we have*

$$\mathrm{p}_g(G) = \begin{cases} \dfrac{1}{p}\left(1 - \dfrac{1}{p^{n-1}}\right) & \textit{if } g \neq 1, \\[2ex] \dfrac{1}{p}\left(1 + \dfrac{p-1}{p^{n-1}}\right) & \textit{if } g = 1. \end{cases}$$

To prove Proposition 3.4 we need the following lemma. Note that in the sequel for a group K and an element k of that group we consider $\overline{K} = K/Z(K)$ and $\overline{k} = kZ(K)$, and define the map $a_K : \overline{K} \times \overline{K} \longrightarrow K'$ by $a_K(\overline{x}, \overline{y}) = [x, y]$ which is obviously well defined.

Lemma 3.5. *Let G and H be two isoclinic finite groups and let (φ, ψ) be an isoclinism from G to H. If $g \in G'$, then we have $\mathrm{p}_g(G) = \mathrm{p}_{\psi(g)}(H)$.*

Proof. Since (φ, ψ) is an isoclinism from G to H, φ is an isomorphism from $\overline{G}$ to $\overline{H}$ and ψ is an isomorphism from G' to H' for which the following diagram commutes.

$$\begin{array}{ccc} \overline{G} \times \overline{G} & \xrightarrow{\varphi \times \varphi} & \overline{H} \times \overline{H} \\ \Big\downarrow a_G & & \Big\downarrow a_H \\ G' & \xrightarrow{\psi} & H' \end{array}$$

We now obtain the following identities:

$$\begin{aligned} |\overline{G}|^2 \mathrm{p}_g(G) &= \frac{1}{|Z(G)|^2} |G|^2 \mathrm{p}_g(G) \\ &= \frac{1}{|Z(G)|^2} |\{(x, y) \in G \times G : [x, y] = g\}| \\ &= \frac{1}{|Z(G)|^2} |\{(x, y) \in G \times G : a_G(\overline{x}, \overline{y}) = g\}| \\ &= |\{(\alpha, \beta) \in \overline{G} \times \overline{G} : a_G(\alpha, \beta) = g\}|. \end{aligned}$$

Since ψ is an isomorphism, the last quantity is equal to

$$|\{(\alpha, \beta) \in \overline{G} \times \overline{G} : \psi(a_G(\alpha, \beta)) = \psi(g)\}|,$$

and commutativity of diagram implies that the above quantity is equal to

$$|\{(\alpha, \beta) \in \overline{G} \times \overline{G} : a_H(\varphi(\alpha), \varphi(\beta)) = \psi(g)\}|.$$

But φ is an isomorphism, so we obtain

$$\begin{aligned} |\overline{G}|^2 \mathrm{p}_g(G) &= |\{(\alpha, \beta) \in \overline{G} \times \overline{G} : a_H(\varphi(\alpha), \varphi(\beta)) = \psi(g)\}| \\ &= |\{(\gamma, \delta) \in \overline{H} \times \overline{H} : a_H(\gamma, \delta) = \psi(g)\}| \\ &= \frac{1}{|Z(H)|^2} |\{(x, y) \in H \times H : a_H(\overline{x}, \overline{y}) = \psi(g)\}| \\ &= \frac{1}{|Z(H)|^2} |\{(x, y) \in H \times H : [x, y] = \psi(g)\}| \\ &= \frac{1}{|Z(H)|^2} |H|^2 \mathrm{p}_{\psi(g)}(H) \\ &= |\overline{H}|^2 \mathrm{p}_{\psi(g)}(H). \end{aligned}$$

But $\overline{G}$ and $\overline{H}$ are isomorphic (via φ), hence $|\overline{G}| = |\overline{H}|$ and the equality $\mathrm{p}_g(G) = \mathrm{p}_{\psi(g)}(H)$ follows as required. □

Proof of Proposition 3.4. Chose an element of $\mathbb{ISO}(G)$, say H. Therefore G and H are isoclinic (say via (φ, ψ)) and we have $p^n = |H|$. Since $G' \leq Z(G)$ and $H \in \mathbb{ISO}(G)$, Lemma 3.3 implies that $H' = Z(H)$. Now by using Lemma 3.5 and Proposition 3.1 we have

$$
\begin{aligned}
\mathrm{p}_g(G) &= \mathrm{p}_{\psi(g)}(H) \\
&= \begin{cases} \dfrac{1}{p}\left(1 - \dfrac{1}{|H : Z(H)|}\right) & \text{if } g \neq 1 \\[2ex] \dfrac{1}{p}\left(1 + \dfrac{p-1}{|H : Z(H)|}\right) & \text{if } g = 1 \end{cases} \\
&= \begin{cases} \dfrac{1}{p}\left(1 - \dfrac{1}{p^{n-1}}\right) & \text{if } g \neq 1 \\[2ex] \dfrac{1}{p}\left(1 + \dfrac{p-1}{p^{n-1}}\right) & \text{if } g = 1 \end{cases}
\end{aligned}
$$

as required. □

4. Groups G which $|G'|$ is prime and $G' \cap Z(G) = 1$

In this section we deal with finite groups G which $|G'|$ is prime and $G' \cap Z(G) = 1$. We give an explicit formula for $\mathrm{p}_g(G)$ for such groups. Following we state an important result which is proved in [12], Proposition 5, that will be useful for further considerations.

Proposition 4.1. *Let G be a finite group such that $|G'| = p$, p prime, and let $Z(G) = 1$. Then $G = \langle a, b : a^p = b^n = 1,\ bab^{-1} = a^r \rangle$, where $n|(p-1)$ and $r^j \equiv 1 \pmod{p}$ if and only if $n|j$.*

Note that the groups in the above proposition are known as *metacyclic* groups. The following proposition gives us a formula for $\mathrm{p}_g(G)$ for these groups.

Proposition 4.2. *Let n and r be positive integers and let p be a prime number for which $n|(p-1)$ and $r^j \equiv 1 \pmod{p}$ if and only if $n|j$. Suppose that $G = \langle a, b : a^p = b^n = 1,\ bab^{-1} = a^r \rangle$. If $g \in G'$, then we have*

$$
\mathrm{p}_g(G) = \begin{cases} \dfrac{n^2 - 1}{pn^2} & if\ g \neq 1, \\[2ex] \dfrac{n^2 + p - 1}{pn^2} & if\ g = 1. \end{cases}
$$

Proof. It is easy to see that $|G'| = p$ and $\mathrm{cd}(G) = \{1, n\}$. Now the assertion holds by Theorem 2.2. □

We need the following lemma which is proved in [7], to find a formula for $\mathrm{p}_g(G)$ for finite groups G which $|G'|$ is prime and $G' \cap Z(G) = 1$.

Lemma 4.3. *Let G be a finite group and N be a normal subgroup of G for which $G' \cap N = 1$. Then G is isoclinic to G/N.*

Now consider a finite group G such that $|G'| = p$ is a prime and $G' \cap Z(G) = 1$. By the above lemma, G is isoclinic to $\overline{G}$. Since $(\overline{G})' = G'Z(G)/Z(G) = (G' \times Z(G))/Z(G) \cong G'$, $\overline{G}$ has derived subgroup of order p. On the other hand, if we consider $Z(\overline{G}) = H/Z(G)$ for some H, $Z(G) \leq H \leq G$, then $[G, H] \leq G' \cap Z(G) = 1$ implying $H = Z(G)$ and we have $Z(\overline{G}) = 1$. Therefore by Proposition 4.1, there is a positive integer n depends only to G for which $\overline{G} = \langle a, b : a^p = b^n = 1,\ bab^{-1} = a^r\rangle$. The number n here called *invariant number* of G and denoted by $\mathrm{i}(G)$.

Finally we can state our formula which we obtained by using definition of invariant number, Lemma 3.5 and Proposition 4.2.

Proposition 4.4. *Let G be a finite group such that $|G'| = p$, p prime, and $G' \cap Z(G) = 1$. Suppose that $\mathrm{i}(G) = n$. If $g \in G'$, then we have*

$$\mathrm{p}_g(G) = \begin{cases} \dfrac{n^2-1}{pn^2} & if\ g \neq 1, \\[2ex] \dfrac{n^2+p-1}{pn^2} & if\ g = 1. \end{cases}$$

5. Upper bounds for $\mathrm{p}_g(G)$

In this section we give upper bounds for $\mathrm{p}_g(G)$. The following proposition gives us an upper bound for this quantity depending only on the number of conjugacy classes of G.

Proposition 5.1. *Let G be a finite group, and let $g \in G'$. Then $\mathrm{p}_g(G) \leq k(G)/|G|$, where $k(G)$ is the number of conjugacy classes of G. Moreover, the equality holds if and only if $g = 1$.*

Proof. By [9, Lemma 2.15(c)] we have $|\chi(g)| \leq \chi(1)$ for each $\chi \in \mathrm{Irr}(G)$. Therefore, Theorem 2.1 now implies that

$$\mathrm{p}_g(G) = \frac{1}{|G|}\sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)} \leq \frac{1}{|G|}\sum_{\chi \in \mathrm{Irr}(G)} \frac{|\chi(g)|}{\chi(1)} \leq \frac{1}{|G|}\sum_{\chi \in \mathrm{Irr}(G)} 1 = \frac{|\mathrm{Irr}(G)|}{|G|} = \frac{k(G)}{|G|}.$$

It is obvious that the equality holds if and only if $|\chi(g)| = \chi(1)$ for each $\chi \in \mathrm{Irr}(G)$ or equivalently $g = 1$. □

Proposition 5.2. *Let G be a finite group, and let g be a nonidentity element of G'. Then we have $\mathrm{p}_g(G) < 1/2$. Also we have $\mathrm{p}_1(G) \leq 5/8$.*

Proof. First, we assume that $g \neq 1$. If $\mathrm{p}_g(G) \geq 1/2$, then by Proposition 5.1 we have $\mathrm{p}_1(G) = k(G)/|G| > \mathrm{p}_g(G) \geq 1/2$ or $\mathrm{p}_1(G) > 1/2$. Therefore we have $|G| = 2^{2s+1}$, $|G'| = 2$ and $\overline{G} \cong {\mathbb{Z}_2}^{2s}$ for some positive integer s. But in this case we have $\mathrm{p}_g(G) = (1 - 1/2^s)/2 < 1/2$ which is a contradiction. Therefore $\mathrm{p}_g(G) < 1/2$ as required. The second part is well known. □

Proposition 5.3. *For any $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there exists a finite group G and $g \in G$ such that $1/2 - \varepsilon < \mathrm{p}_g(G) < 1/2$.*

Proof. Let s be a positive integer such that $s > -\log_2 2\varepsilon$. Consider a finite group G of order 2^{2s+1} such that $|G'| = 2$ and $\overline{G} \cong {\mathbb{Z}_2}^{2s}$ and let g be a nonidentity element of G'. Then $1/2 - \varepsilon < \mathrm{p}_g(G) = (1 - 1/2^s)/2 < 1/2$ as required. □

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