

*PROBABILITY THAT AN ELEMENT OF A FINITE GROUP  
HAS A SQUARE ROOT*

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**Abstract.** Let  $G$  be a finite group of even order. We give some bounds for the probability  $p(G)$  that a randomly chosen element in  $G$  has a square root. In particular, we prove that  $p(G) \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|$ . Moreover, we show that if the Sylow 2-subgroup of  $G$  is not a proper normal elementary abelian subgroup of  $G$ , then  $p(G) \leq 1 - 1/\sqrt{|G|}$ . Both of these bounds are best possible upper bounds for  $p(G)$ , depending only on the order of  $G$ .

**1. Introduction.** Let  $G$  be a finite group and let  $g \in G$ . If there exists an element  $h \in G$  for which  $g = h^2$ , then we say that  $g$  has a *square root*. Clearly,  $g$  may have one or more square roots, or it may have none. Let  $G^2$  be the set of all elements of  $G$  which have at least one square root, i.e.,

$$G^2 = \{g \in G \mid \text{there exists } h \in G \text{ such that } g = h^2\},$$

or simply  $G^2 = \{g^2 \mid g \in G\}$ . Then

$$p(G) = \frac{|G^2|}{|G|}$$

is the probability that a randomly chosen element in  $G$  has a square root.

The properties of  $p(S_n)$ , where  $S_n$  denotes the symmetric group on  $n$  letters, have been studied by some authors. Asymptotic properties of  $p(S_n)$  were studied in [1], [2], [8] and in [3], which is devoted to the proof of a conjecture of Wilf [9] that  $p(S_n)$  is non-increasing in  $n$ . Recently, the basic properties of  $p(G)$  for an arbitrary finite group  $G$  have been studied by the authors of this paper (see [7]). Moreover, they calculated  $p(G)$  when  $G$  is a simple group of Lie type of rank 1 or when  $G$  is an alternating group. A table of  $p(G)$  for the sporadic finite simple groups was also given.

In this paper we give some bounds for the probability that a randomly chosen element in a given finite group has a square root. In particular, we

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give the following best possible upper bounds for  $p(G)$ , depending only on  $|G|$  (see Theorems 2.11 and 2.13).

**MAIN THEOREM.** *Let  $G$  be a finite group of even order. Then*

$$p(G) \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|.$$

*Moreover, if the Sylow 2-subgroup of  $G$  is not a proper normal elementary abelian subgroup of  $G$ , then  $p(G) \leq 1 - 1/\sqrt{|G|}$ , and both bounds are the best possible.*

**2. The best possible bounds.** By [7, Proposition 2.1(ii)],  $p(G) = 1$  if and only if  $|G|$  is odd. Therefore we deal with even order groups. The following theorem presents an upper bound for  $p(G)$  when  $G$  has even order, improving the bound  $p(G) < 1$ .

**THEOREM 2.1.** *Let  $G$  be a finite group of even order, and  $P$  be a Sylow 2-subgroup of  $G$ . Then  $p(G) \leq 1 - 1/|P|$ .*

Let  $P$  be the additive group of the field  $\text{GF}(2^n)$  and let  $H = \text{GF}(2^n)^\times$  be its multiplicative group. Let  $G = PH$  be the semidirect product of these groups, with  $H$  acting on  $P$  by multiplication. Then  $p(G) = 1 - 1/|P|$ , which shows that the bound in Theorem 2.1 is sharp.

The following corollary is just a combination of Theorem 2.1 and Proposition 2.3 of [7].

**COROLLARY 2.2.** *Let  $G$  be a finite group of even order, and  $P$  be a Sylow 2-subgroup of  $G$ . If  $G$  is solvable, then  $1/|P| \leq p(G) \leq 1 - 1/|P|$ .*

We recall that if a Sylow 2-subgroup of a finite group is cyclic, then the group has a normal 2-complement (see for example [6, 7.2.2]), and it is therefore solvable. We thus get the following corollary.

**COROLLARY 2.3.** *Let  $G$  be a finite group such that  $|G| = 2m$ , where  $m$  is odd. Then  $p(G) = 1/2$ .*

In order to prove Theorem 2.1, we must first explain a few things about decomposition of an element in a finite group. So let  $G$  be a finite group. We can uniquely decompose each element  $x \in G$  into  $x = x_2 x_{2'} = x_{2'} x_2$ , where  $x_2$  is a 2-element of  $G$  and  $x_{2'}$  is an element of  $G$  of odd order. Moreover, if  $x$  has a square root then so also does  $x_2$ . In the following, when we speak about  $x_2$  and  $x_{2'}$ , we always mean this unique decomposition of  $x$ . We also need the following result originally proved by Frobenius (see [5] and also Corollary 41.11 of [4] as a more accessible reference).

**REMARK 2.4.** Let  $G$  be a finite group,  $a \in G$ , and  $n$  be a positive integer. Then the number of solutions of the equation  $x^n = a$  in  $G$  is a multiple of  $\text{gcd}(n, |C_G(a)|)$ . In particular, the number of solutions of the equation  $x^n = 1$  in  $G$  is a multiple of  $\text{gcd}(n, |G|)$ .

*Proof of Theorem 2.1.* Choose  $a \in G$  such that  $a$  is a 2-element of maximal order in  $G$ . We claim that if  $x \in G$  and  $x = x_2x_{2'}$  with  $x_2$  a conjugate of  $a$ , then  $x$  does not have a square root. To prove the claim, suppose that  $a = h^2$  for some  $h \in G$ . Then by [7, Remark 2.2] we have  $|h| = 2|a|$ , which contradicts the definition of  $a$ . Therefore  $a$  does not have a square root and the same is true for its conjugates. Hence,  $x_2$  does not have a square root, which in turn implies that  $x$  does not have a square root. Therefore the claim holds and we have

$$\{x \in G \mid x_2 \text{ is conjugate to } a\} \subseteq G \setminus G^2.$$

Observe also that the number of  $x \in G$  for which  $x_2$  is conjugate to  $a$  is equal to  $|G : C_G(a)|t$ , where  $t$  is the number of elements of odd order of  $C_G(a)$ . Therefore

$$|G : C_G(a)|t \leq |G| - |G^2|.$$

We now write  $|G| = 2^k m$  where  $k \geq 1$  and  $m$  is odd. Then it is clear that  $|C_G(a)| = 2^{k'} m'$  for some positive integers  $k'$  and  $m'$  such that  $k' \leq k$  and  $m' \mid m$ . On the other hand, it is easy to see that an element  $x$  in  $C_G(a)$  has odd order if and only if  $x^{m'} = 1$ . Therefore,  $t$  is equal to the number of solutions of the equation  $x^{m'} = 1$  in  $C_G(a)$ . By Remark 2.4, this is a multiple of  $\gcd(m', 2^{k'} m') = m'$ . Hence,  $m' \leq t$  and thus  $|G : C_G(a)|m' \leq |G : C_G(a)|t \leq |G| - |G^2|$ . By dividing both sides by  $|G|$  we obtain

$$\frac{m'}{|C_G(a)|} \leq 1 - p(G),$$

which in turn implies that

$$p(G) \leq 1 - \frac{m'}{2^{k'} m'} = 1 - \frac{1}{2^{k'}} \leq 1 - \frac{1}{2^k} = 1 - \frac{1}{|P|},$$

as required. ■

The following theorem gives another upper bound for  $p(G)$  when  $G$  has even order, depending only on the order of  $G$  and the number of 2-elements of  $G$ .

**THEOREM 2.5.** *Let  $G$  be a finite group of even order, and denote by  $Q$  the set of 2-elements of  $G$ . Then  $p(G) \leq 1 - |Q|/2|G|$ .*

*Proof.* Suppose  $a \in Q$ . By Remark 2.4, the number of solutions of the equation  $x^2 = a$  in  $G$  is a multiple of  $\gcd(2, |C_G(a)|)$ . Hence, this number is either 0 or  $\geq 2$ . But by [7, Remark 2.2] all solutions of this equation lie in  $Q$ . Therefore,  $|G| - |G^2| \geq |Q|/2$ , or  $p(G) \leq 1 - |Q|/2|G|$  as required. ■

We now prove an easy but useful lemma.

**LEMMA 2.6.** *Let  $G$  be a finite group, and  $N$  be a normal subgroup of  $G$ . Then  $p(G) \leq p(G/N)$ .*

*Proof.* Note that  $gN \in G/N$  has a square root if and only if there is  $xN \in G/N$  for which  $gN = (xN)^2$  if and only if  $x^2 \in gN$ . Therefore,  $gN \in G/N$  does not have a square root if and only if there is no element  $x \in G$  with  $x^2 \in gN$ . Hence, if a coset in  $G/N$  does not have a square root, then no element of this coset has a square root in  $G$ , and therefore  $|G| - |G^2| \geq |N|(|G/N| - |(G/N)^2|)$ . By dividing both sides by  $|G|$  we obtain  $1 - p(G) \geq 1 - p(G/N)$ , or  $p(G) \leq p(G/N)$  as required. ■

As corollaries of Lemma 2.6, we give an upper bound for  $p(G)$  when  $G$  is a finite 2-group, depending only on the order of  $G$ , and then an upper bound for  $p(G)$  when  $G$  is a finite nilpotent group.

**COROLLARY 2.7.** *Let  $G$  be a finite 2-group such that  $|G| \geq 4$ . Then  $p(G) \leq 1 - 1/\sqrt{|G|}$ .*

*Proof.* Suppose that  $\Phi(G)$  is the Frattini subgroup of  $G$ . By Lemma 2.6 and Theorem 2.4(i) of [7], we have

$$p(G) \leq p\left(\frac{G}{\Phi(G)}\right) = \frac{1}{|G/\Phi(G)|} \leq \frac{1}{2}.$$

Since  $|G| \geq 4$ , we obtain  $1/2 \leq 1 - 1/\sqrt{|G|}$ , and so the above inequality implies that  $p(G) \leq 1 - 1/\sqrt{|G|}$  as required. ■

**COROLLARY 2.8.** *Let  $G$  be a finite nilpotent group of even order, and  $P$  be a Sylow 2-subgroup of  $G$ . If  $|P| = 2$ , then  $p(G) = 1/2$ . If  $|P| > 2$ , then  $1/|P| \leq p(G) \leq 1 - 1/\sqrt{|P|} \leq 1 - 1/\sqrt{|G|}$ .*

*Proof.* The first statement is Corollary 2.3. The second statement comes from Corollary 2.7 and Proposition 2.3 of [7], which states that if  $G$  is nilpotent, then  $p(G) = p(P)$ . ■

The following two propositions give upper bounds for  $p(G)$ , depending on the order of  $G$ , but only for special classes of even order groups.

**PROPOSITION 2.9.** *Let  $G$  be a finite group of even order. If  $G$  contains more than one Sylow 2-subgroup, then  $p(G) \leq 1 - 1/\sqrt{|G|}$ .*

*Proof.* Let  $P$  be a Sylow 2-subgroup of  $G$ . Since  $G$  has at least two distinct Sylow 2-subgroups,  $P$  is not normal in  $G$ . By Remark 2.4, the number of solutions of the equation  $x^{|P|} = 1$  in  $G$  is a multiple of  $\gcd(|P|, |G|) = |P|$ . Therefore,  $|P|$  divides the number of solutions of  $x^{|P|} = 1$  in  $G$ . But if we let  $Q$  be the set of 2-elements of  $G$ , then the set of solutions of the equation  $x^{|P|} = 1$  in  $G$  is just  $Q$ , and this means  $|P|$  divides  $|Q|$ . Hence, either  $|P| = |Q|$  or  $|P| \leq |Q|/2$ . In the first case  $P = Q$  is normal in  $G$ , contrary to hypothesis. Hence,  $|P| \leq |Q|/2$ . On the other hand, by Theorem 2.5, we have  $p(G) \leq 1 - |Q|/2|G|$ , and so  $p(G) \leq 1 - |P|/|G|$ . This inequality together

with Theorem 2.1 now implies that  $(1 - p(G))^2 \geq (|P|/|G|)(1/|P|) = 1/|G|$ , and so  $p(G) \leq 1 - 1/\sqrt{|G|}$  as required. ■

**PROPOSITION 2.10.** *Let  $G$  be a finite group of even order with elementary abelian Sylow 2-subgroups. Then  $p(G) \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|$ .*

*Proof.* Suppose  $P$  is an elementary abelian Sylow 2-subgroup of  $G$ . Consider  $x \neq 1$  as an element of  $P$ . If there is  $y \in G$  such that  $x = y^2$ , then by [7, Remark 2.2] we have  $|y| = 4$ , which is a contradiction. Therefore,  $x \in G \setminus G^2$ , and so  $P \setminus \{1\} \subseteq G \setminus G^2$ . Hence,  $|P| - 1 \leq |G| - |G^2|$ . On the other hand, by Theorem 2.1,  $p(G) \leq 1 - 1/|P|$  and so  $|G^2| \leq |G| - |G|/|P|$ , which implies  $|G|/|P| \leq |G| - |G^2|$ . Therefore,  $|G| - |G|/|P| \leq (|G| - |G^2|)^2$ , or

$$|G| \leq (|G| - |G^2|)^2 + |G|/|P| \leq (|G| - |G^2|)(|G| - |G^2| + 1) < (|G| - |G^2| + 1)^2.$$

This implies that  $\sqrt{|G|} < |G| - |G^2| + 1$ , so  $\lfloor \sqrt{|G|} \rfloor \leq |G| - |G^2|$ , and hence  $p(G) \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|$  as required. ■

The bound of Proposition 2.10 is the best possible. In fact, if  $G$  is the group described just after the statement of Theorem 2.1, then  $p(G) = 1 - \lfloor \sqrt{|G|} \rfloor / |G|$ .

We can now state the following theorem which gives lower and upper bounds for  $p(G)$ , depending only on the order of  $G$ .

**THEOREM 2.11.** *Let  $G$  be a finite group of even order. Then*

$$1/|G| \leq p(G) \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|.$$

*Proof.* It is clear that  $1/|G| \leq p(G)$  (see also Proposition 2.1 of [7]). Therefore we prove the second inequality. We first consider groups  $G$  with  $|G| < 26$ . Among these, by Corollary 2.3, we only need to deal with groups whose order is divisible by 4. Moreover, if  $G$  is nilpotent, then by Proposition 2.3 of [7] and by Corollary 2.7, we have

$$p(G) = p(P) \leq 1 - \frac{1}{\sqrt{|P|}} \leq 1 - \frac{1}{\sqrt{|G|}} \leq 1 - \frac{\lfloor \sqrt{|G|} \rfloor}{|G|},$$

and we are done. Therefore we should prove the second inequality only for groups of order 12, 20 and 24. In these cases, if the Sylow 2-subgroup is normal, we are done, and otherwise we can use Proposition 2.9. Hence, the second inequality holds for groups  $G$  with  $|G| < 26$ .

We now suppose that  $|G| \geq 26$ . Let  $N \neq 1$  be a minimal normal subgroup of  $G$ .

Suppose that  $G/N$  has odd order. In this case  $|N|$  is even. Since  $N$  is minimal normal, it is isomorphic to a direct product of isomorphic simple groups. There are two possibilities. If  $N \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$  is an elementary abelian 2-group, then  $N$  is the unique Sylow 2-subgroup of  $G$ . Hence,

Proposition 2.10 implies that  $p(G) \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|$ , which gives the second inequality. If  $N \cong S \times \cdots \times S$ , where  $S$  is a non-abelian simple group, then  $G$  has at least two distinct Sylow 2-subgroups and so, by Proposition 2.9, we obtain  $p(G) \leq 1 - 1/\sqrt{|G|} \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|$ , which gives the second inequality.

Next we assume that  $G/N$  has even order. In this case, we apply induction on  $|G|$ . Since  $|G/N| < |G|$ , the inductive hypothesis implies that

$$(1) \quad p(G/N) \leq 1 - \frac{\lfloor \sqrt{|G/N|} \rfloor}{|G/N|},$$

and therefore, by Lemma 2.6, we have

$$(2) \quad p(G) \leq 1 - \frac{\lfloor \sqrt{|G/N|} \rfloor}{|G/N|}.$$

We claim that if  $|N| \geq 12$ , then

$$(3) \quad 1 - \frac{\lfloor \sqrt{|G/N|} \rfloor}{|G/N|} \leq 1 - \frac{\lfloor \sqrt{|G|} \rfloor}{|G|}.$$

To prove the claim, observe that (3) is equivalent to  $\lfloor \sqrt{|G|} \rfloor \leq \lfloor \sqrt{|G/N|} \rfloor |N|$ . Therefore it is enough to prove that  $\sqrt{|G|} \leq (\sqrt{|G/N|} - 1)|N|$ , that is,  $\sqrt{|G|} \geq |N|/(\sqrt{|N|} - 1)$ . Since  $|G| \geq 2|N|$ , it is sufficient to show that  $\sqrt{2} \geq \sqrt{|N|}/(\sqrt{|N|} - 1)$ , which is true for  $|N| \geq 12$ . Therefore the claim holds and so for  $|N| \geq 12$  we get, using (2), the inequality  $p(G) \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|$ , which is the second inequality.

We now suppose that  $|N| \leq 11$ . We observe that (1) is equivalent to

$$|G/N| - |(G/N)^2| \geq \lfloor \sqrt{|G/N|} \rfloor.$$

Therefore there are at least  $\lfloor \sqrt{|G/N|} \rfloor$  cosets  $g_1N, \dots, g_lN$  such that there is no  $x \in G$  with  $x^2 \in g_iN$ ,  $i = 1, \dots, l$ . Consequently,

$$(4) \quad |G| - |G^2| \geq |N| \lfloor \sqrt{|G/N|} \rfloor.$$

For any  $N$  such that  $1 < |N| \leq 11$ , it is easy to prove that

$$\frac{|N|}{\sqrt{|N|} - 1} < 5.$$

Since  $|G| \geq 26$ , we have  $\sqrt{|G|} > 5$ , therefore

$$\frac{|N|}{\sqrt{|N|} - 1} < 5 < \sqrt{|G|}.$$

This implies that  $|N| < \sqrt{|G|}(\sqrt{|N|} - 1)$ , which can be rewritten as

$$0 < \sqrt{|G|}\sqrt{|N|} - \sqrt{|G|} - |N|,$$

or

$$0 < \sqrt{|N|}(\sqrt{|G|} - \sqrt{|N|}) - \sqrt{|G|}.$$

So we have

$$\sqrt{|G|} < |N|(\sqrt{|G/N|} - 1) < |N|\lfloor\sqrt{|G/N|}\rfloor.$$

Since  $\lfloor\sqrt{|G|}\rfloor \leq \sqrt{|G|}$ , using (4) we get  $\lfloor\sqrt{|G|}\rfloor \leq |G| - |G^2|$ , which gives  $p(G) \leq 1 - \lfloor\sqrt{|G|}\rfloor/|G|$ . ■

The cyclic group of order 4 shows that the bound in Theorem 2.11 is the best possible. In fact,

$$p(\mathbb{Z}_4) = 1/2 = 1 - 1/\sqrt{4}.$$

A natural question arises: *Does the slightly stronger bound of Proposition 2.9 hold if  $P$  is normal but  $\Phi(P) > 1$ , so that only elementary abelian normal Sylow 2-subgroups are responsible for the weaker bound of Theorem 2.11?*

The answer is yes, as we prove in the following theorem.

**THEOREM 2.12.** *Let  $G$  be a finite group of even order, and  $P$  be a Sylow 2-subgroup of  $G$ . If  $p(G) > 1 - 1/\sqrt{|G|}$ , then  $P$  is a proper normal elementary abelian subgroup of  $G$ .*

*Proof.* By Proposition 2.9,  $P$  is normal, and by Corollary 2.8,  $G$  is not nilpotent and therefore  $P \neq G$ . Let  $\Phi = \Phi(P)$  be the Frattini subgroup of  $P$ . We first suppose that  $\sqrt{|G|} \leq |P|/2$ . Then  $1/\sqrt{|G|} \leq |P|/2|G|$ , which implies, by Theorem 2.5,

$$p(G) \leq 1 - \frac{|P|}{2|G|} \leq 1 - \frac{1}{\sqrt{|G|}},$$

contrary to hypothesis.

Therefore we can suppose that  $|\Phi|^2 \leq |P|^2/4 \leq |G|$ . Then, by Lemma 2.6 and Theorem 2.11, we have

$$p(G) \leq p(G/\Phi) \leq 1 - \frac{\lfloor\sqrt{|G/\Phi|}\rfloor}{|G/\Phi|} \leq 1 - \frac{|\Phi|(\sqrt{|G/\Phi|} - 1)}{|G|}.$$

We want to prove that

$$\frac{|\Phi|(\sqrt{|G/\Phi|} - 1)}{|G|} \geq \frac{1}{\sqrt{|G|}}.$$

This is equivalent to showing that

$$(5) \quad \sqrt{|G|} \geq \frac{|\Phi|}{\sqrt{|\Phi|} - 1}.$$

We first suppose that  $|\Phi| \geq 4$ ; then  $\sqrt{|\Phi|} - 1 \geq 1$  and the inequality (5) is equivalent to  $|\Phi|^2 \leq |G|$ , which we are assuming is true.

We then suppose  $|\Phi| = 2$ . If  $P$  is cyclic, then by the remark preceding Corollary 2.3,  $P$  has a normal 2-complement  $Q$ . Hence  $G = P \times Q$  and by Corollary 2.7,

$$\begin{aligned} p(G) &= p(P \times Q) = p(P)p(Q) = p(P) \\ &\leq 1 - \frac{1}{\sqrt{|P|}} \leq 1 - \frac{1}{\sqrt{|G|}}, \end{aligned}$$

contrary to hypothesis. Thus  $P$  is not cyclic, and this implies  $|P| \geq 8$  and  $|G| \geq 24$ , so again

$$\sqrt{|G|} \geq \sqrt{24} > \frac{2}{\sqrt{2}-1} = \frac{|\Phi|}{\sqrt{|\Phi|}-1},$$

which is (5).

Thus (5) holds in both cases, and this implies  $p(G) \leq 1 - 1/\sqrt{|G|}$ , contrary to hypothesis. This last contradiction proves that  $\Phi = \{1\}$ . ■

We close this section by observing that Theorems 2.11 and 2.12 together prove the following theorem. Moreover, the group  $G$  described just after the statement of Theorem 2.1 shows that the bound  $p(G) \leq 1 - \lfloor \sqrt{|G|} \rfloor / |G|$  in Theorem 2.11 is the best possible and the cyclic group of order 4 shows that the better bound  $p(G) \leq 1 - 1/\sqrt{|G|}$  is again the best possible.

**THEOREM 2.13.** *Let  $G$  be a finite group of even order. If the Sylow 2-subgroup of  $G$  is not a proper normal elementary abelian subgroup of  $G$ , then*

$$p(G) \leq 1 - 1/\sqrt{|G|}.$$

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