RINGS WHICH ARE GENERATED BY THEIR UNITS: A GRAPH THEORETICAL APPROACH

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Dedicated to the memory of Maria Silvia Lucido

Abstract. The ring $\mathbb{Z}_2 \times \mathbb{Z}_2$, having only one unit, cannot be generated by its units. It turns out, in the general theory of rings, that this is essentially the only example. In this note, we give an elementary proof of "A finite commutative ring with nonzero identity is generated by its units if and only if it cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient." The proof uses graph theory, and offers, as a byproduct, that in this case, every element is the sum of at most three units.

1. Introduction

The study of rings which are generated additively by their units seems to have arisen in 1953–1954 when Wolfson [13] and Zelinsky [14] proved, independently, that if V is a finite or infinite dimensional vector space over a division ring D, then every linear transformation is the sum of two nonsingular linear transformations unless $\dim V = 1$ and $D = \mathbb{Z}_2$. This implies that the ring of linear transformations $\mathrm{End}_D(V)$ is generated additively by its units. In fact, every element of $\mathrm{End}_D(V)$ is the sum of two units except for one obvious case when V is a one dimensional vector space over $\mathbb{Z}_2$. Wolfson's and Zelinsky's result caused quite a bit of interest in the study of rings that are generated by their units.

In 1958, Skornyakov [8, Page 167, Problem 31], posed the problem of determining which regular rings are generated by their units. More precisely, he asked: *Is every element of a von Neumann regular ring, which cannot have $\mathbb{Z}_2$ as a quotient, a sum of units?* – This question of Skornyakov was answered negatively by Bergman in 1977 (see [5] which is a significant contribution to the theory of von Neumann regular rings). Bergman constructed a von Neumann regular algebra in which not all elements are sums of units.

In 1968, while apparently unaware of Skornyakov's book, Ehrlich [2] produced a large class of regular rings generated by their units. He proved that if R is a ring such that 2 is a unit and for every $a \in R$ there exists a unit $u \in R$ such that $aua = a$, then every element of R is the sum of two units.

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In 1974, Raphael [7] launched a systematic study of rings generated by their units, which he calls S-rings.

Finally, in 1976, Fisher and Snider [3] proved that if R is a von Neumann regular ring with primitive factor rings artinian and 2 is a unit, then every element of R can be expressed as the sum of two units.

In 1998, Wolfson's and Zelinsky's result was reproved by Goldsmith, Pabst and Scott where they remarked that this result can hardly be new but they were unable to find any reference to it in the literature (see [4]). Interest in this topic increased recently after they defined the unit sum number in [4].

For additional historical background the reader is referred to the paper [10], which also contains references to recent work in this area. Also see [9] for a survey of rings which are generated by their units.

The purpose of this note is to give an elementary proof of Theorem 1.1. The proof uses graph theory, and offers, as a byproduct, that if R is a finite commutative ring with nonzero identity which is generated by its units, then every element of R can be written as a sum of at most three units.

Theorem 1.1 ([7], Corollary 7). *Let R be a finite commutative ring with nonzero identity. Then R is generated by its units if and only if R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.*

2. Basic notation and properties of graphs

In this section we introduce some notation and definitions of graphs that will be used throughout the note. We also state and prove Lemmas 2.1 and 2.2 which are required in Section 3. Here, by a graph G we mean a finite undirected graph without loops or multiple edges (unless otherwise specified). The reader is referred to [1] and [12] for a fuller treatment of the subject.

For a graph G, let $V(G)$ denote the set of vertices. Let G be a graph and suppose $x, y \in V(G)$. We recall that a *walk* between x and y is a sequence $x = v_0, e_1, v_1, \ldots, e_k, v_k = y$ of vertices and edges of G, denoted by

$$x = v_0 \xrightarrow{e_1} v_1 \longrightarrow \cdots \xrightarrow{e_k} v_k = y,$$

such that for every i with $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. Also a *path* between x and y is a walk between x and y without repeated vertices. The number of edges in a walk (counting repeats) or a path is called its *length*.

For the proof of Theorem 1.1 we need the following well-known fact. We state and prove it here for the convenience of the reader.

Lemma 2.1. *Let x and y be distinct vertices of a graph G. If there is a walk between x and y then there is also a path between x and y.*

Proof. By assumption there is a walk between x and y and so we may select a walk

$$W : x = v_0 \xrightarrow{e_1} v_1 \longrightarrow \cdots \xrightarrow{e_k} v_k = y$$

of minimal length k between x and y. If W is not a path, select a vertex that appears twice, say $v_i = v_j$ where $i < j$. Consider

$$W' : x = v_0 \xrightarrow{e_1} v_1 \longrightarrow \cdots \xrightarrow{e_i} v_i \xrightarrow{e_{j+1}} v_{j+1} \longrightarrow \cdots \xrightarrow{e_k} v_k = y.$$

Then W' is a walk between x and y with length shorter than k, a contradiction. Therefore W is a path between x and y. $\square$

A graph G is called *connected* if for all vertices x and y there exists a path between x and y. Otherwise, G is called *disconnected*.

A *bipartite* graph is one whose vertex-set is partitioned into two (not necessarily nonempty) disjoint subsets, called *parts*, in such a way that the two end vertices for each edge lie in distinct parts. Among bipartite graphs, a *complete bipartite* graph is one in which each vertex is joined to every vertex that is not in the same part.

Let G_1 and G_2 be two vertex-disjoint graphs. The *category product* of G_1 and G_2 is denoted by $G_1 \dot{\times} G_2$. That is, $V(G_1 \dot{\times} G_2) := V(G_1) \times V(G_2)$; two distinct vertices (x, y) and (x', y') are adjacent if and only if x is adjacent to x' in G_1 and y is adjacent to y' in G_2.

We now state and prove the following lemma which will be used in the proof of Theorem 1.1. A bipartite graph is *nontrivial* if both parts of its vertex-set are nonempty. For more information on this lemma we refer the reader to [11].

Lemma 2.2. *Let G_1 and G_2 be two bipartite graphs at least one of which is nontrivial. Then $G_1 \dot{\times} G_2$ is disconnected.*

Proof. We assume that G_2 is nontrivial. Thus G_1 is partitioned into two disjoint subsets X_1 and Y_1 as well as G_2 into two disjoint subsets X_2 and Y_2 in such a way that $|X_1| \geq 1$, $|X_2| \geq 1$ and $|Y_2| \geq 1$. Choose $a \in X_1$, $b \in X_2$ and $c \in Y_2$. We claim that there is no path between (a, b) and (a, c) in $G_1 \dot{\times} G_2$. In order to do this, suppose in contrary that, there is a path P between (a, b) and (a, c) in $G_1 \dot{\times} G_2$:

$$P : \quad (a, b) \xrightarrow{e_1} (a_1, b_1) \xrightarrow{e_2} (a_2, b_2) \longrightarrow \cdots \xrightarrow{e_{n-1}} (a_{n-1}, b_{n-1}) \xrightarrow{e_n} (a, c).$$

We now obtain the walk $\hat{W}$ in G_1 and the walk $\tilde{W}$ in G_2 both with length n:

$$\hat{W} : \quad a \xrightarrow{\hat{e}_1} a_1 \xrightarrow{\hat{e}_2} a_2 \longrightarrow \cdots \xrightarrow{\hat{e}_{n-1}} a_{n-1} \xrightarrow{\hat{e}_n} a,$$

$$\tilde{W} : \quad b \xrightarrow{\tilde{e}_1} b_1 \xrightarrow{\tilde{e}_2} b_2 \longrightarrow \cdots \xrightarrow{\tilde{e}_{n-1}} b_{n-1} \xrightarrow{\tilde{e}_n} c.$$

The existence of the walk $\hat{W}$ implies that n is even while the existence of the walk $\tilde{W}$ implies that n is odd, a contradiction. Thus there is no path between (a, b) and (a, c) in $G_1 \dot{\times} G_2$, which implies that $G_1 \dot{\times} G_2$ is disconnected. $\square$

Let us consider yet a few more definitions required for a complete understanding of the next section. For a graph G and vertices x and y of G, the *distance* between x and y, denoted by $d(x, y)$, is the number of edges in a shortest path between x and y. If there is no path between x and y then we write $d(x, y) = \infty$. We recall that the largest distance among all distances between pairs of the vertices of a graph G

is called the *diameter* of G and is denoted by $\operatorname{diam}(G)$. Finally, for a given vertex $x \in V(G)$, the *neighbor set* of x is the set $N_G(x) := \{v \in V(G) \mid v \text{ is adjacent to } x\}$. Moreover, if G has a loop at vertex x, then we always assume that $x \in N_G(x)$.

3. Proof of Theorem 1.1

In this section, using the results presented in Section 2, we are able to prove Theorem 1.1.

Let R be an arbitrary finite associative ring R with nonzero identity, say 1, which is preserved by homomorphisms and inherited by subrings. Let U_R be the set of units of R. We attach a graph to R, denoted by G_R, based on the elements and units of R. This graph is obtained by letting R be the set of vertices and defining distinct vertices x and y to be adjacent if and only if $x + y \in U_R$. If we omit the word "distinct" in the definition of G_R, we obtain the graph $\overline{G}_R$; this graph may have loops. Note that if $2 \notin U_R$, then $\overline{G}_R = G_R$.

The graphs in Fig. 1 are the graphs attached to the rings indicated.

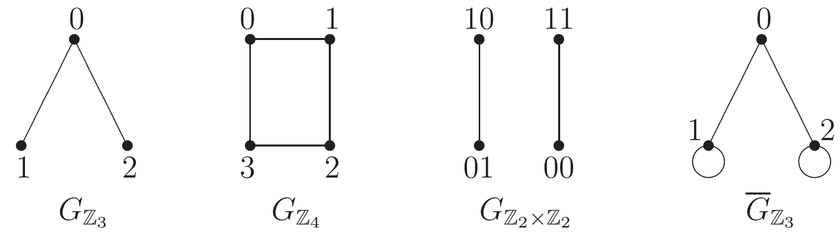

Figure 1.

It is easy to see that, for given rings R and S, if $R \cong S$ as rings, then $G_R \cong G_S$ as graphs. Also we have $\overline{G}_R \dot{\times} \overline{G}_S \cong G_{R\times S}$.

In Fig. 2 we illustrate these points for the direct product of the rings $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

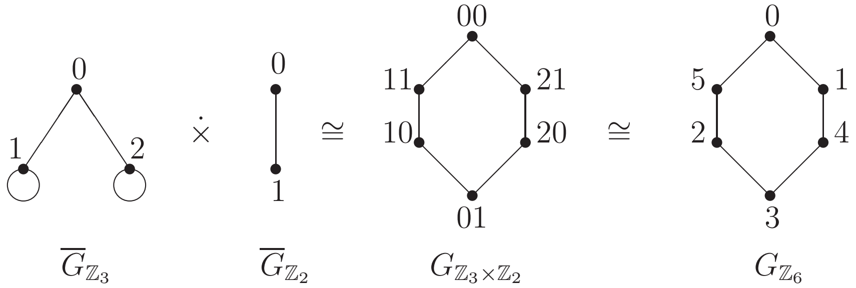

Figure 2.

We need the following result, which is useful in the sequel.

Lemma 3.1. *Let R be a finite commutative local ring with maximal ideal $\mathfrak{m}$. Then the following statements hold:*

(a) *If* $|R/\mathfrak{m}| = 2$, *then* G_R *is a complete bipartite graph.*
(b) *If* $|R/\mathfrak{m}| > 2$, *then for every* $x, y \in R$ *we have* $N_{\overline{G}_R}(x) \cap N_{\overline{G}_R}(y) \neq \emptyset$.

Proof. Part (a): Let $X = \mathfrak{m}$ and $Y = R \setminus \mathfrak{m}$. We have $V(G_R) = X \cup Y$ and $X \cap Y = \emptyset$. Therefore X and Y partition $V(G_R)$ into two subsets. It is clear that no pair of distinct elements of X are adjacent. We show that no distinct elements of Y are adjacent. In order to do this, fix an element in $R \setminus \mathfrak{m}$, say a. By assumption we have $R = \mathfrak{m} \cup (\mathfrak{m} + a) = \mathfrak{m} \cup (\mathfrak{m} + (-a))$. Now for distinct elements x and y in $R \setminus \mathfrak{m}$, we may write $x = m + a$ and $y = m' - a$ where $m, m' \in \mathfrak{m}$. If $x + y \in U_R$, then we conclude that $m + m' \in U_R$. Therefore $\mathfrak{m}$ has a unit element and so $\mathfrak{m} = R$, a contradiction. Thus $x + y \notin U_R$, which implies that x and y are not adjacent. Therefore no distinct elements of Y are adjacent. Hence G_R is a bipartite graph.

Suppose that $x \in X$ and $y \in Y$ are given. If $x + y \notin U_R$, then $x + y \in X$ and so $y \in X$, a contradiction. Thus $x + y \in U_R$, which implies that x and y are adjacent. Therefore each vertex of X is joined to every vertex of Y and so G_R is complete bipartite.

Part (b): By assumption we conclude that $|U_R| \geq 2|R|/3$. Suppose that x is an arbitrary element of R and fix it. There are two possibilities: either $2x \notin U_R$ or $2x \in U_R$. If $2x \notin U_R$, then $\overline{G}_R$ has no loop at vertex x. On the other hand, for every element $u - x$, where $u \in U_R$, we have $u - x \neq x$ and $u - x$ is adjacent to x in $\overline{G}_R$. This implies that $\{u - x \mid u \in U_R\} \subseteq N_{\overline{G}_R}(x)$ and so $|N_{\overline{G}_R}(x)| \geq |U_R| \geq 2|R|/3$. If $2x \in U_R$, then $\overline{G}_R$ has a loop at vertex x. On the other hand, for every element $u - x$, where $u \in U_R \setminus \{2x\}$, we have $u - x \neq x$ and $u - x$ is adjacent to x in $\overline{G}_R$. This implies that $\{u - x \mid u \in U_R \setminus \{2x\}\} \cup \{x\} = \{u - x \mid u \in U_R\} \subseteq N_{\overline{G}_R}(x)$ and so we have again $|N_{\overline{G}_R}(x)| \geq |U_R| \geq 2|R|/3$. Therefore, in both cases, we have $|N_{\overline{G}_R}(x)| \geq 2|R|/3$.

Now, for every $x, y \in R$,

$$\begin{aligned}
|N_{\overline{G}_R}(x) \cap N_{\overline{G}_R}(y)| &= |N_{\overline{G}_R}(x)| + |N_{\overline{G}_R}(y)| - |N_{\overline{G}_R}(x) \cup N_{\overline{G}_R}(y)| \\
&\geq (2|R|/3) + (2|R|/3) - |R| \\
&= |R|/3 \\
&> 0
\end{aligned}$$

and so $N_{\overline{G}_R}(x) \cap N_{\overline{G}_R}(y) \neq \emptyset$ as required. □

Now let R be a finite commutative ring with nonzero identity and fix it. We want to prove R is generated by its units if and only if R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. We start with the proposition below which contains a necessary and sufficient condition for G_R to be connected.

Proposition 3.2. G_R *is connected if and only if* R *is generated by its units.*

Proof. ($\Longrightarrow$) Suppose that $a \in R$ is written by the sum of some units and $b \in R$ is adjacent to a in G_R. Therefore $a + b \in U_R$ and so we may write $b = c - a$, for some $c \in U_R$. Thus b is the sum of some units.

Now suppose that $x \in R$ is given. Since G_R is connected, there exists a path between x and 1 and, therefore, by the above observation we conclude that x is the sum of some units. This means that R is generated by its units.

($\Longleftarrow$) Suppose that $a \in R$. Since R is generated by its units, we may write $a = u_1 + \cdots + u_k$, where $u_i \in U_R$, $1 \leq i \leq k$. We now have the walk

$$0 \xrightarrow{e_1} -u_1 \xrightarrow{e_2} u_1 + u_2 \xrightarrow{e_3} -u_1 - u_2 - u_3 \xrightarrow{e_4} u_1 + u_2 + u_3 + u_4 \longrightarrow \cdots \xrightarrow{e_k} u_1 + \cdots + u_k = a$$

between 0 and a, when k is even and the walk

$$0 \xrightarrow{e_1} u_1 \xrightarrow{e_2} -u_1 - u_2 \xrightarrow{e_3} u_1 + u_2 + u_3 \xrightarrow{e_4} -u_1 - u_2 - u_3 - u_4 \longrightarrow \cdots \xrightarrow{e_k} u_1 + \cdots + u_k = a$$

between 0 and a, when k is odd.

This implies that for every $x, y \in R$ there is a walk W_1 between x and 0 as well as a walk W_2 between 0 and y. The walks W_1 and W_2 together form a walk W between x and y. By using Lemma 2.1, we conclude that there is also a path P between x and y, which implies the connectedness of G_R. $\square$

The following proposition contains another necessary and sufficient condition for G_R to be connected.

Proposition 3.3. *G_R is connected if and only if R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.*

Proof. Every finite commutative ring with nonzero identity is isomorphic to a direct product of finite local rings (see [6, Page 95]). Therefore, we may write $R \cong R_1 \times \cdots \times R_k$, where every R_i is a local ring with maximal ideal $\mathfrak{m}_i$.

($\Longrightarrow$) Suppose by contrary that R has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. This implies that for at least two i, for example $i = 1, 2$, we have $|R_i/\mathfrak{m}_i| = 2$. Now part (a) of Lemma 3.1 implies that G_{R_1} and G_{R_2} are both bipartite. Thus by using Lemma 2.2, we conclude that $G_{R_1} \dot{\times} G_{R_2}$ is disconnected.

On the other hand, by the observation just before Fig. 2, we have

$$G_R \cong \begin{cases} \overline{G}_{R_1} \dot{\times} \overline{G}_{R_2} & \text{if } k = 2, \\ (\overline{G}_{R_1} \dot{\times} \overline{G}_{R_2}) \dot{\times} \overline{G}_{R_3 \times \cdots \times R_k} & \text{if } k \geq 3. \end{cases}$$

But for $i = 1, 2$ we have $2 \notin U_{R_i}$ and so $\overline{G}_{R_i} = G_{R_i}$. Therefore we obtain

$$G_R \cong \begin{cases} G_{R_1} \dot{\times} G_{R_2} & \text{if } k = 2, \\ (G_{R_1} \dot{\times} G_{R_2}) \dot{\times} \overline{G}_{R_3 \times \cdots \times R_k} & \text{if } k \geq 3. \end{cases}$$

Now the disconnectedness of $G_{R_1} \dot{\times} G_{R_2}$ implies that G_R is also disconnected. This contradiction shows that R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.

($\Longleftarrow$) By assumption R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. This implies that for at most one i, we have $|R_i/\mathfrak{m}_i| = 2$. There are the following cases to be considered:

(1) *$|R_i/\mathfrak{m}_i| > 2$ holds for every i.*

Suppose that $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ are arbitrary distinct elements of $R_1 \times \cdots \times R_k$. Since for every i with $1 \leq i \leq k$ we have $|R_i/\mathfrak{m}_i| > 2$, by using part (b) of Lemma 3.1 we conclude that $N_{\overline{G}_{R_i}}(x_i) \cap N_{\overline{G}_{R_i}}(y_i) \neq \emptyset$. Therefore we may choose $z_i \in N_{\overline{G}_{R_i}}(x_i) \cap N_{\overline{G}_{R_i}}(y_i)$. Thus we have the following walk in $\overline{G}_{R_1\times\cdots\times R_k}$:

$$(x_1, \dots, x_k) \xrightarrow{e_1} (z_1, \dots, z_k) \xrightarrow{e_2} (y_1, \dots, y_k).$$

This implies that $d(x, y) \leq 2$ and so $\operatorname{diam}(G_R) = \operatorname{diam}(G_{R_1\times\cdots\times R_k}) \leq 2$.

(2) *$|R_i/\mathfrak{m}_i| > 2$ holds for every i except one of them.*

First, suppose that $k = 1$. In this case $R \cong R_1$ is a finite local ring with maximal ideal $\mathfrak{m}_1$ in such a way $|R_1/\mathfrak{m}_1| = 2$. Thus, if R is a field, then we have $R \cong \mathbb{Z}_2$ and so $\operatorname{diam}(G_R) = 1$. If R is not a field, then by using part (a) of Lemma 3.1 we conclude that G_R is complete bipartite with $|R| \geq 4$ and so $\operatorname{diam}(G_R) = 2$.

Second, suppose that $k \geq 2$. In this case we may assume that $|R_1/\mathfrak{m}_1| = 2$ and $|R_i/\mathfrak{m}_i| > 2$ for every i with $2 \leq i \leq k$. Suppose that $x = (x_1, x_2, \dots, x_k)$ and $y = (y_1, y_2, \dots, y_k)$ are arbitrary distinct elements of $R_1 \times R_2 \times \cdots \times R_k$. If either $x_1, y_1 \in \mathfrak{m}_1$ or $x_1, y_1 \notin \mathfrak{m}_1$, then by the same argument as above, we obtain a path between x and y with length at most 2. This implies that $d(x, y) \leq 2$. Now, we may assume that $x_1 \in \mathfrak{m}_1$ and $y_1 \notin \mathfrak{m}_1$. For every i with $2 \leq i \leq k$, consider w_i as follows:

$$w_i = \begin{cases} 1 & \text{if } x_i \in \mathfrak{m}_i, \\ 0 & \text{if } x_i \notin \mathfrak{m}_i. \end{cases}$$

On the other hand, since for every i with $2 \leq i \leq k$ we have $|R_i/\mathfrak{m}_i| > 2$, by using part (b) of Lemma 3.1 we conclude that $N_{\overline{G}_{R_i}}(w_i) \cap N_{\overline{G}_{R_i}}(y_i) \neq \emptyset$. Therefore we may choose $z_i \in N_{\overline{G}_{R_i}}(w_i) \cap N_{\overline{G}_{R_i}}(y_i)$. Thus we have the following walk in $\overline{G}_{R_1\times R_2\times\cdots\times R_k}$:

$$(x_1, x_2, \dots, x_k) \xrightarrow{e_1} (y_1, w_2, \dots, w_k) \xrightarrow{e_2} (x_1, z_2, \dots, z_k) \xrightarrow{e_3} (y_1, y_2, \dots, y_k).$$

This implies that $d(x, y) \leq 3$. Therefore, for every distinct $x, y \in R_1 \times R_2 \times \cdots \times R_k$ we have $d(x, y) \leq 3$ and so $\operatorname{diam}(G_R) = \operatorname{diam}(G_{R_1\times R_2\times\cdots\times R_k}) \leq 3$.

Therefore in both cases we have $\operatorname{diam}(G_R) \leq 3$. Thus every two vertices of G_R are joined by a path with length at most 3, which implies that G_R is connected. □

Propositions 3.2 and 3.3 imply that R is generated by its units if and only if R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient which completes the proof of Theorem 1.1. Our proof shows that if R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, then not only G_R is connected, but also $\operatorname{diam}(G_R) \leq 3$. Therefore, we may state

Corollary 3.4. *Let R be a finite commutative ring with nonzero identity. If R is generated by its units, or equivalently, R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, then every element of R can be written as a sum of at most three units.*

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