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UNIT GRAPHS ASSOCIATED WITH RINGS

N. Ashrafi[1], H. R. Maimani[2,5], M. R. Pournaki[3,5], and S. Yassemi[4]
[1]*Department of Mathematics, Semnan University, Semnan, Iran*
[2]*Mathematics Section, Department of Basic Sciences, Shahid Rajaee Teacher Training University, Tehran, Iran*
[3]*Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran*
[4]*School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran*
[5]*School of Mathematics, Institute for Research in Fundamental Sciences (IPM), Tehran, Iran*

Let R be a ring with nonzero identity. The unit graph of R, denoted by $G(R)$, has its set of vertices equal to the set of all elements of R; distinct vertices x and y are adjacent if and only if $x + y$ is a unit of R. In this article, the basic properties of $G(R)$ are investigated and some characterization results regarding connectedness, chromatic index, diameter, girth, and planarity of $G(R)$ are given. (These terms are defined in Definitions and Remarks 4.1, 5.1, 5.3, 5.9, and 5.13.)

Key Words: Chromatic index; Connectedness; Diameter; Girth; Planarity; Unit graph.

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1. INTRODUCTION

Let n be a positive integer, and let $\mathbb{Z}_n$ be the ring of integers modulo n. Ralph P. Grimaldi defined a graph $G(\mathbb{Z}_n)$ based on the elements and units of $\mathbb{Z}_n$ (cf. [7]). The *vertices* of $G(\mathbb{Z}_n)$ are the elements of $\mathbb{Z}_n$; distinct vertices x and y are defined to be *adjacent* if and only if $x + y$ is a unit of $\mathbb{Z}_n$. That is, xy is an *edge* if and only if $x + y$ is a unit. For a positive integer m, it follows that $G(\mathbb{Z}_{2m})$ is a $\varphi(2m)$-regular graph, where φ is the Euler phi function (cf. Definitions 2.3). In case $m \geq 2$, $G(\mathbb{Z}_{2m})$ can be expressed as the union of $\varphi(2m)/2$ *Hamiltonian cycles*, i.e., cycles containing all the vertices of the graph (cf. Definitions and Remarks 3.1). The odd case is not quite so easy, but the structure is clear and the results are similar to the even case. When p is an odd prime, $G(\mathbb{Z}_p)$ can be expressed as a cone over a complete partite graph with $(p-1)/2$ partitions of size two (cf. Definitions and Remarks 2.6). This leads to an explicit formula for the *chromatic polynomial* $p(k)$ of $G(\mathbb{Z}_p)$. Here, $p(k)$

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Address correspondence to M. R. Pournaki, Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran; Fax: +98 21 6600 5117. E-mail: pournaki@ipm.ir

is the number of ways needed to color the vertices of $G(\mathbb{Z}_p)$ using at most k colors in such a way that no two adjacent vertices receive the same color. The chromatic polynomial of a graph was introduced by Birkhoff in 1912 in order to study the 4-color problem (cf. [2]). The article [7] also concludes with some properties of the graphs $G(\mathbb{Z}_{p^m})$, where p is a prime number and $m \geq 2$.

We now generalize $G(\mathbb{Z}_n)$ to $G(R)$, the *unit graph* of R, where R is an arbitrary associative ring with nonzero identity. This graph is a subgraph of the "comaximal" graph studied in [12]. The definition of $G(R)$ and its basic properties are given in the next section. In Section 3, we give some characterizations of unit graphs. In Section 4, we give a necessary and sufficient condition for $G(R)$ to be connected; connectedness is related to the unit sum number (cf. Definitions and Remarks 4.2). We define and study chromatic index, diameter, girth, and the planarity of $G(R)$ in Section 5.

In what follows, all rings are associative with nonzero identity, denoted by 1, which is preserved by ring homomorphisms and inherited by subrings. Also throughout the article, by a graph G we mean a finite undirected graph without loops or multiple edges (unless otherwise specified). We use the notation of Kaplansky [9] and Chartrand and Oellermann [3].

2. BASIC NOTATION AND PROPERTIES

In this section we study the basic properties of unit graphs. First we define the unit graph and the closed unit graph corresponding to an associative ring.

Definitions and Remarks 2.1. Let R be a ring and $U(R)$ be the set of unit elements of R. The *unit graph* of R, denoted $G(R)$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be *adjacent* if and only if $x + y \in U(R)$. If we omit the word "distinct" in the definition, we obtain the *closed unit graph* denoted $\overline{G}(R)$; this graph may have loops. Note that if $2 \notin U(R)$, then $\overline{G}(R) = G(R)$.

The graphs in Fig. 1 are the unit graphs of the rings indicated.

It is easy to see that, for given rings R and S, if $R \cong S$ as rings, then $G(R) \cong G(S)$ as graphs. This point is illustrated in Fig. 2, for the unit graphs of two isomorphic rings $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$.

Next we give additional basic notation and define graph products.

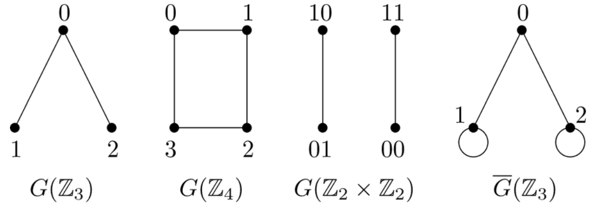

Figure 1 The unit graphs of some specific rings.

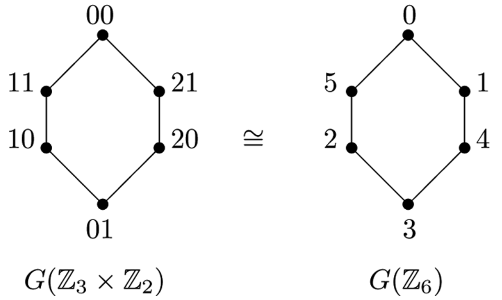

Figure 2 The unit graphs of two isomorphic rings.

Definitions and Remarks 2.2. For a graph G, let $V(G)$ denote the set of vertices, and let $E(G)$ denote the set of edges. Let G_1 and G_2 be two vertex-disjoint graphs. The *category product* of G_1 and G_2 is denoted $G_1 \dot{\times} G_2$. That is, $V(G_1 \dot{\times} G_2) := V(G_1) \times V(G_2)$; two distinct vertices (x, y) and (x', y') are *adjacent* if and only if x is adjacent to x' in G_1 and y is adjacent to y' in G_2. Clearly, for given rings R_1 and R_2, two distinct vertices $(x, y), (x', y') \in V(\overline{G}(R_1) \dot{\times} \overline{G}(R_2))$ are *adjacent* if and only if x is adjacent to x' in $\overline{G}(R_1)$ and y is adjacent to y' in $\overline{G}(R_2)$. This implies that $\overline{G}(R_1) \dot{\times} \overline{G}(R_2) \cong G(R_1 \times R_2)$.

In Fig. 3, we illustrate the above point for the direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

We now state some basic properties of unit graphs. First we give more definitions.

Definitions 2.3. For a graph G and vertex $x \in V(G)$, the *degree* of x, denoted $\deg(x)$, is the number of edges of G incident with x. For every nonnegative integer r, the graph G is called *r-regular* if the degree of each vertex of G is equal to r. Also, for a given vertex $x \in V(G)$, the *neighbor set* of x is the set $N_G(x) := \{v \in V(G) \mid v \text{ is adjacent to } x\}$. Moreover, if G has a *loop* at vertex x, then we always assume that $x \in N_G(x)$. The *closed neighbor set* of x, is the set $N_G[x] := N_G(x) \cup \{x\}$.

Proposition 2.4 shows that, if 2 is not a unit element in the ring R, then the unit graph of R is a $|U(R)|$-regular graph (cf. Definitions 2.3). The unit graph is *not* $|U(R)|$-regular when 2 is a unit of R.

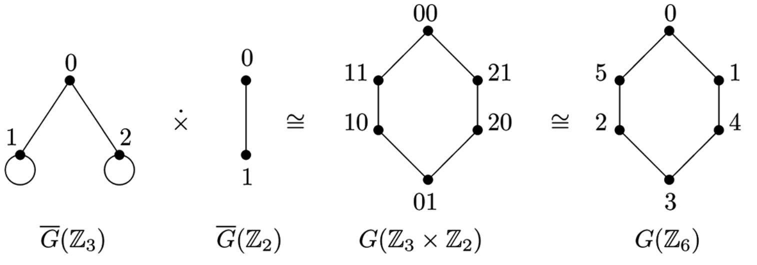

Figure 3 The category product of two closed rings.

Proposition 2.4. *Let R be a finite ring. Then the following statements hold for the unit graph of R:*

(a) *If $2 \notin U(R)$, then the unit graph $G(R)$ is a $|U(R)|$-regular graph;*
(b) *If $2 \in U(R)$, then for every $x \in U(R)$ we have $\deg(x) = |U(R)| - 1$ and for every $x \in R \backslash U(R)$ we have $\deg(x) = |U(R)|$.*

Proof. For the proof of both (a) and (b), suppose that the vertex $x \in R$ is given. We have $R + x = R$, and so for every $u \in U(R)$ there exists an element $x_u \in R$ such that $x_u + x = u$. Clearly, x_u is uniquely determined by u.

First, suppose that $2 \notin U(R)$. In this case $x_u \neq x$ and so x_u is adjacent to x in $G(R)$. Therefore, $f : U(R) \longrightarrow N_{G(R)}(x)$ given by $f(u) = x_u$ is a well-defined function. It is easy to see that f is a bijection and, therefore, $\deg(x) = |N_{G(R)}(x)| = |U(R)|$. This shows that, in this case, the unit graph $G(R)$ is a $|U(R)|$-regular graph. This completes the proof of (a).

Second, suppose that $2 \in U(R)$ and $x \in R \backslash U(R)$. In this case, we have again $x_u \neq x$, and so x_u is adjacent to x in $G(R)$. Therefore, the above observation is still valid, which shows that $\deg(x) = |U(R)|$.

Finally, suppose that $2 \in U(R)$ and $x \in U(R)$. In this case, $2x \in U(R)$ and we have $x_u \neq x$ for $u \neq 2x$ and $x_{2x} = x$. Now x_u is adjacent to x in $G(R)$ for $u \neq 2x$. Therefore, $f : U(R) \longrightarrow N_{G(R)}[x]$ given by $f(u) = x_u$ is a well-defined function. It is easy to see that f is a bijection and, therefore, $\deg(x) = |N_{G(R)}(x)| = |N_{G(R)}[x]| - 1 = |U(R)| - 1$. Thus (b) holds. □

Remark 2.5. Let R be a finite local ring and let $\mathfrak{m}$ be a maximal ideal of R. If $|R/\mathfrak{m}| > 2$, then $|U(R)| \geq 2|R|/3$. Thus for every $x, y \in R$, by using the proof of Proposition 2.4, we may write

$$\begin{aligned} |N_{\overline{G}(R)}(x) \cap N_{\overline{G}(R)}(y)| &= |N_{\overline{G}(R)}(x)| + |N_{\overline{G}(R)}(y)| - |N_{\overline{G}(R)}(x) \cup N_{\overline{G}(R)}(y)| \\ &\geq |U(R)| + |U(R)| - |R| \geq |R|/3 > 0. \end{aligned}$$

This implies that $N_{\overline{G}(R)}(x) \cap N_{\overline{G}(R)}(y) \neq \emptyset$. We will use this result in the proof of Lemma 5.5.

Definitions and Remarks 2.6. A *complete graph* is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. For r a nonnegative integer, an *r-partite graph* is one whose vertex-set is partitioned into r disjoint parts in such a way that the two end vertices for each edge lie in distinct partitions. A *complete r-partite graph* is one in which each *vertex* is joined to every vertex that is not in the same partition. The *complete* 2-*partite graph* (also called the *complete bipartite graph*) with exactly two partitions of size m and n, is denoted by $K_{m,n}$. A *clique* of a graph G is a complete subgraph of G. A *coclique* in a graph G is a set of pairwise nonadjacent vertices. A subgraph H of a graph G is called a *spanning subgraph* if $V(H) = V(G)$. A 1-regular spanning subgraph H of G is called a *perfect matching* of G. For a graph G and a nonempty subset $S \subseteq V(G)$, the *vertex-induced subgraph*, denoted $\langle S \rangle$, is the subgraph of G with vertex-set S and edges incident to members of S. Note that subgraphs may be induced by sets of edges. Indeed, for a graph G and a nonempty subset $A \subseteq E(G)$, the *edge-induced subgraph*, denoted $\langle A \rangle$, is the subgraph of G with edge-set A and vertex-set consists of all ends of edges of A. We also point out that if $\{G_i\}_{i\in I}$ is a family of edge-disjoint subgraphs of a graph G such that $E(G) = \bigcup_{i\in I} E(G_i)$, then we may write $G = \bigoplus_{i\in I} G_i$. In this case, if $G_i \cong H$ for every $i \in I$, then we write $G = \bigoplus_I H$.

We need the following lemma to prove Proposition 2.8.

Lemma 2.7. *Let R be a commutative ring and suppose that $J(R)$ denotes the Jacobson radical of R. If $x, y \in R$, then the following statements hold:*

(a) *If $x + J(R)$ and $y + J(R)$ are adjacent in $G(R/J(R))$, then every element of $x + J(R)$ is adjacent to every element of $y + J(R)$ in $G(R)$;*
(b) *If $2x \notin U(R)$, then $x + J(R)$ is a coclique in $G(R)$;*
(c) *If $2x \in U(R)$, then $x + J(R)$ is a clique in $G(R)$.*

Proof. For part (a), suppose that $a \in x + J(R)$ and $b \in y + J(R)$ are given. We may write $a = x + j$ and $b = y + j'$, where $j, j' \in J(R)$. Since $x + J(R)$ and $y + J(R)$ are adjacent in $G(R/J(R))$, there exists a unit element $u \in U(R)$ in such a way that $(x + J(R)) + (y + J(R)) = u + J(R)$. Thus $x + y - u = (a + b) - (j + j') - u \in J(R)$. Now suppose, on the contrary, that $a + b$ is not a unit element in R. Therefore, $\langle a + b \rangle$ is a proper ideal of R, and so there exists a maximal ideal $\mathfrak{m}$ of R such that $\langle a + b \rangle \subseteq \mathfrak{m}$. Thus $a + b \in \mathfrak{m}$. On the other hand, $j + j'$ and $(a + b) - (j + j') - u$ are elements of $\mathfrak{m}$ and so $u \in \mathfrak{m}$, a contradiction. Thus $a + b \in U(R)$, and therefore, a and b are adjacent in $G(R)$.

For part (b), suppose that two elements of $x + J(R)$, say z and z', are adjacent in $G(R)$. Therefore, $z + z' \in U(R)$. We may write $z = x + j$ and $z' = x + j'$, where $j, j' \in J(R)$. Thus $2x + j + j' \in U(R)$. By the assumption, $2x \notin U(R)$, and so there exists a maximal ideal $\mathfrak{m}$ of R such that $\langle 2x \rangle \subseteq \mathfrak{m}$. Therefore, $2x \in \mathfrak{m}$ and since $j + j' \in \mathfrak{m}$, we obtain that $2x + j + j' \in \mathfrak{m}$. Therefore, $\mathfrak{m}$ has a unit element, a contradiction. This means that $x + J(R)$ is a coclique in $G(R)$.

For part (c), suppose that z and z' are arbitrary elements of $x + J(R)$. We may write $z = x + j$ and $z' = x + j'$, where $j, j' \in J(R)$. If $z + z'$ is not a unit element of R, then there exists a maximal ideal $\mathfrak{m}$ of R such that $\langle z + z' \rangle \subseteq \mathfrak{m}$. Thus $z + z' = 2x + j + j' \in \mathfrak{m}$ and since $j + j' \in \mathfrak{m}$, we obtain that $2x \in \mathfrak{m}$, a contradiction with the assumption that $2x \in U(R)$. Thus $z + z' \in U(R)$ and so z and z' are adjacent in $G(R)$. This means that $x + J(R)$ is a clique in $G(R)$. □

Proposition 2.8. *Let R be a finite commutative ring and suppose that $J(R)$ denotes the Jacobson radical of R. Then the following statements hold:*

(a) *If $2 \notin U(R)$, then $G(R) \cong \bigoplus_{J(R)^2} G(R/J(R))$;*
(b) *If $2 \in U(R)$, then $G(R) \cong (\bigoplus_{J(R)^2} G(R/J(R))) \oplus (\bigoplus_{U(R)} K_{|J(R)|})$.*

Proof. Clearly, we may partition R into subsets $x_i + J(R)$, $i \in I$, where $\{x_i\}_{i\in I}$ is a set of representatives of the distinct cosets of $J(R)$ in $R/J(R)$. Also, by part (a) of Lemma 2.7, for every $i, j \in I$, if $x_i + J(R)$ and $x_j + J(R)$ are adjacent in $G(R/J(R))$, then every element of $x_i + J(R)$ is adjacent to every element of $x_j + J(R)$ in $G(R)$.

Now, for part (a), the assumption $2 \notin U(R)$ implies that for every $i \in I$, $2x_i \notin U(R)$ and so, by part (b) of Lemma 2.7, $x_i + J(R)$ is a coclique in $G(R)$. Therefore, $G(R) \cong \bigoplus_{J(R)^2} G(R/J(R))$.

For part (b), the assumption $2 \in U(R)$ implies that $2x_i \in U(R)$, for every $i \in I$ with $x_i \in U(R)$. Therefore, by part (c) of Lemma 2.7, $x_i + J(R)$ is a clique in $G(R)$, for every $i \in I$ with $x_i \in U(R)$. Thus comparing with part (a), we conclude that $G(R) \cong (\bigoplus_{J(R)^2} G(R/J(R))) \oplus (\bigoplus_{U(R)} K_{|J(R)|})$. □

Our next result involves the Nagata extension of a ring, and so we recall this notion here. Let R be a commutative ring and let M be a unitary R-module. We may turn the cartesian product $R \times M$ into a ring by defining the operations $(r, x) + (r', x') := (r + r', x + x')$ and $(r, x)(r', x') := (rr', rx' + r'x)$. This ring is called the *Nagata extension* of R by M and is denoted by $R \oplus M$. Clearly $(1, 0)$ is the identity of $R \oplus M$.

The following result gives information about the unit graph of the Nagata extension of a ring.

Proposition 2.9. *Let R be a commutative ring and let M be a finite unitary R-module. Then the following statements hold*:

(a) *If $2 \notin U(R)$, then $G(R \oplus M) \cong \bigoplus_{M^2} G(R)$;*
(b) *If $2 \in U(R)$, then $G(R \oplus M) \cong (\bigoplus_{M^2} G(R)) \oplus (\bigoplus_{U(R)} K_{|M|})$.*

Proof. First, for every $r \in R$, we define $W_r = \{(r, x) \mid x \in M\}$. Clearly, for distinct elements r and r' of R, $W_r \cap W_{r'} = \emptyset$, and we have $R \oplus M = \bigcup_{r \in R} W_r$. Therefore, we may partition $R \oplus M$ into subsets W_r, $r \in R$. One easily verifies that $(r, x) \in R \oplus M$ is a unit if and only if $r \in R$ is a unit. Therefore, if $(r, x) \in W_r$ is adjacent to $(r', x') \in W_{r'}$ in $G(R \oplus M)$, then r is adjacent to r' in $G(R)$. In addition, if r is adjacent to r' in $G(R)$, then every element of W_r is adjacent to every element of $W_{r'}$ in $G(R \oplus M)$. Thus for every $e \in E(G(R))$, the unit graph $G(R \oplus M)$ has a subgraph, say H^e, which is isomorphic to $K_{|M|,|M|}$. On the other hand, by [3, p. 192], we have $K_{|M|,|M|} = \bigoplus_{i \in M} M_i^{(e)}$, where each of $M_i^{(e)}$ is a perfect matching of $K_{|M|,|M|}$. We consider $K_i := \bigoplus_{e \in E(G(R))} M_i^{(e)}$, $i \in M$, which is a subgraph of $G(R \oplus M)$.

Now, for part (a), the assumption $2 \notin U(R)$ implies that for every $r \in R$, W_r is a coclique in $G(R \oplus M)$. Therefore, we may write $G(R \oplus M) \cong \bigoplus_{i \in M} K_i$. If for $e \in E(G(R))$ we choose $x_i^{(e)} \in E(M_i^{(e)})$, then the edge-induced subgraph $\langle x_i^{(e)} \rangle$ of $G(R)$ is isomorphic to $G(R)$. Since every perfect matching $M_i^{(e)}$ contains $|M|$ edges, we obtain that $K_i \cong \bigoplus_{i \in M} G(R)$ and so we have $G(R \oplus M) \cong \bigoplus_{x \in M}(\bigoplus_{i \in M} G(R)) \cong \bigoplus_{M^2} G(R)$.

Finally, for part (b), it is easy to see that the assumption $2 \in U(R)$ implies that for every $r \in U(R)$, W_r is a clique in $G(R \oplus M)$, and so comparing with part (a), we conclude that $G(R \oplus M) \cong (\bigoplus_{M^2} G(R)) \oplus (\bigoplus_{U(R)} K_{|M|})$. □

3. CHARACTERIZATIONS OF CERTAIN UNIT GRAPHS

In this section, we study more properties and consequences of the unit graphs.

Definitions and Remarks 3.1. Let G be graph, and suppose $x, y \in V(G)$. We recall that a *walk* between x and y is a sequence $x = v_0, e_1, v_1, \ldots, e_k, v_k = y$ of vertices and edges of G, denoted by

$$x = v_0 \xrightarrow{e_1} v_1 \longrightarrow \cdots \xrightarrow{e_k} v_k = y,$$

such that for every i with $1 \le i \le k$, the edge e_i has endpoints v_{i-1} and v_i. Also a *path* between x and y is a walk between x and y without repeated vertices. A *cycle* of a graph is a path such that the start and end vertices are the same. Two cycles are considered the same if they consist of the same vertices and edges. The number of edges (counting repeats) in a walk, path or a cycle, is called its *length*. A *cycle graph* is a graph that consists of a single cycle. We denote the cycle graph with n vertices by C_n. If a graph G is not bipartite then G contains a cycle of odd length. This result was found by the Hungarian mathematician, Dénes König (1884–1944), in 1916 (cf. [10]). His celebrated textbook *Theorie der endlichen und unendlichen Graphen* appeared in 1936, and was the first book to present graph theory as a subject in its own right (cf. [11]). Note that the converse of König's result is also true, and so the bipartite graphs are characterized by the absence of cycles of odd length.

The following result characterizes the unit graphs of rings that are cycle graphs.

Theorem 3.2. *Let R be a finite ring. Then the unit graph $G(R)$ is a cycle graph if and only if R is isomorphic to one of the following rings*:

(a) $\mathbb{Z}_4$;
(b) $\mathbb{Z}_6$;
(c) $\left\{ \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2 \right\}$.

Proof. Suppose that the unit graph $G(R)$ is a cycle graph. By applying Corollary 4.5 of [5] we obtain that R is isomorphic to one of the following rings:

$R_1 = \mathbb{Z}_3$;
$R_2 = \mathbb{Z}_4$;
$R_3 = \left\{ \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2 \right\}$;
$R_4 = \left\{ \left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] \mid a, b, c \in \mathbb{Z}_2 \right\}$;
$S_i = \mathbb{Z}_2 \times R_i, \ 1 \le i \le 4$.

Among these rings only the unit graphs of R_2, R_3, and S_1 are cycle graphs: $G(R_2) \cong C_4 \cong G(R_3)$ and $G(S_1) \cong C_6$. This completes the proof of the theorem. □

Remark 3.3. By applying Corollary 4.5 of [5] we obtain that $G(R)$ is a 2-regular graph if and only if $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times S$, where S is one of the rings appeared in the statement of Theorem 3.2.

The following result characterizes the unit graphs of rings that are complete graphs (cf. Definitions and Remarks 2.6). Here the characteristic of the ring R is denoted by $\mathrm{Char}(R)$.

Theorem 3.4. *Let R be a ring. Then the unit graph $G(R)$ is a complete graph if and only if R is a division ring with* $\mathrm{Char}(R) = 2$.

Proof. Suppose that the unit graph $G(R)$ is a complete graph. Let $r \neq 0$ be given in R. Since the zero of R is adjacent to r, we have $r = r + 0 \in U(R)$. Therefore, every nonzero element of R is a unit and so R is a division ring. On the other hand, 1 is not adjacent to -1, and so we have $1 = -1$ which implies that $\mathrm{Char}(R) = 2$.

Conversely, if R is a division ring with $\mathrm{Char}(R) = 2$, then each pair of distinct elements of R is adjacent, and so the unit graph $G(R)$ is a complete graph. □

We now state Theorem 3.5, which characterizes the complete bipartite unit graphs of rings (cf. Definitions and Remarks 2.6).

Theorem 3.5. *Let R be a commutative ring and $\mathfrak{m}$ be a maximal ideal of R such that $|R/\mathfrak{m}| = 2$. Then $G(R)$ is a bipartite graph. Moreover, the unit graph $G(R)$ is a complete bipartite graph if and only if R is a local ring.*

Proof. Let $V_1 = \mathfrak{m}$ and $V_2 = R\backslash\mathfrak{m}$. We have $V(G(R)) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Therefore V_1 and V_2 partition $V(G(R))$ into two subsets. It is clear that no pair of distinct elements of V_1 are adjacent. Thus to prove the first part it is enough to show that no elements of V_2 are adjacent. In order to do this, fix an element in $R\backslash\mathfrak{m}$, say a. By assumption, we have $R = \mathfrak{m} \cup (\mathfrak{m} + a) = \mathfrak{m} \cup (\mathfrak{m} + (-a))$. Now for distinct elements x and y in $R\backslash\mathfrak{m}$, we may write $x = m + a$ and $y = m' - a$ where $m, m' \in \mathfrak{m}$. If $x + y \in U(R)$, then we conclude that $m + m' \in U(R)$. Therefore, $\mathfrak{m}$ has a unit element and hence $\mathfrak{m} = R$, a contradiction. Thus $x + y \notin U(R)$, which implies that x and y are not adjacent. Therefore, no elements of V_2 are adjacent. Hence $G(R)$ is a bipartite graph.

Moreover, suppose that R is a local ring with $\mathfrak{m}$ as the unique maximal ideal. Let $x \in V_1$ and $y \in V_2$ be given. If $x + y \notin U(R)$, then $x + y \in V_1$ and so $y \in V_1$, a contradiction. Thus $x + y \in U(R)$, which implies that x and y are adjacent. Therefore, each vertex of V_1 is joined to every vertex of V_2, and so $G(R)$ is complete bipartite.

Finally, we suppose that the unit graph $G(R)$ is a complete bipartite graph. Let V_1 and V_2 be two parts of $G(R)$, and suppose that $0 \in V_1$. This easily implies that $V_2 = U(R)$. We claim that $V_1 = J(R)$, the Jacobson radical of R. Clearly, $J(R) \subseteq V_1$. Now suppose that $x \in V_1$ is given. Then x is not a unit, and so for every $a \in R$, $-ax$ is not a unit; that is, $-ax \in V_1$. Thus $1 - ax \in U(R)$, for every $a \in R$. This implies that $x \in J(R)$, and therefore $V_1 \subseteq J(R)$. Hence we have $V_1 = J(R) = R\backslash U(R)$, which implies that R is a local ring. □

Remarks 3.6. A ring R is called *clean* if every element of R is the sum of an idempotent and a unit, and R is called *uniquely clean* if this representation is unique. This definition dates back to 1977 where it was introduced by Nicholson [14]. In [15] the authors proved that the ring R is uniquely clean if and only if for each maximal ideal $\mathfrak{m}$ of R, $|R/\mathfrak{m}| = 2$. By combining this observation with Theorem 3.5, we conclude that if, for example, R is one of the following rings then the unit graph $G(R)$ is bipartite.

$R = \mathbb{Z}_n$ for even n.
R is a uniquely clean ring.
$R = S \times S'$ where S is a uniquely clean ring and S' is an arbitrary ring.

Also note that if n is an integer such that $n = 2^m$, then $G(\mathbb{Z}_n)$ is a complete bipartite graph.

Definitions 3.7. Let G_1 and G_2 be two graphs. The *join* of G_1 and G_2, which is denoted by $G_1 \vee G_2$, is a graph with vertex-set $V(G_1) \cup V(G_2)$ and edge-set $E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1), y \in V(G_2)\}$. We recall that if we omit a perfect matching (cf. Definitions and Remarks 2.6) from a complete graph K_{2n}, then the resulting graph is called a *cocktail party* and denoted by $\mathrm{CP}(2n)$.

We now state our next result, which gives us some information on the structure of the unit graphs of fields.

Theorem 3.8. *Let R be a ring, n be a positive integer, and let p be an odd prime. Then $G(R) \cong K_1 \vee \mathrm{CP}(p^n - 1)$ if and only if R is a field with p^n elements.*

Proof. Suppose that $G(R) \cong K_1 \vee \mathrm{CP}(p^n - 1)$. Hence there exists an element $r \in R$ which is adjacent to every element of R. Since $G(R)$ is not regular, every element of R except r is invertible and hence $2 \in U(R)$ and $r = 0$. Therefore, R is a field with p^n elements.

Conversely, it is easy to see that if in the unit graph $G(R)$, we remove the zero, then the remaining graph is a complete graph without the edge $\{-r, r\}$ for every $r \neq 0$. Hence this latter graph is a cocktail party and so it is isomorphic to $\mathrm{CP}(p^n - 1)$. Since the zero of R is adjacent to each nonzero elements of R, we conclude that $G(R) \cong K_1 \vee \mathrm{CP}(p^n - 1)$. □

Note that in the above theorem, if R is a field with $\mathrm{Char}(R) \neq 2$, then the unit graph $G(R)$ is isomorphic to $K_1 \vee H$, where H is an infinite cocktail party.

4. CONNECTEDNESS OF UNIT GRAPHS

In this section we study the connectedness of the unit graph $G(R)$. In particular, we give a necessary and sufficient condition for $G(R)$ to be connected, based on the unit sum number of R (cf. Definitions and Remarks 4.2 and Theorem 4.3).

Definitions and Remarks 4.1. A graph G is called *connected* if for any vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected*. The maximal connected subgraphs of G are its *connected components*. Here, maximal means that including any more vertices would yield a disconnected subgraph. Any graph is a *union* of its connected components. If the number of connected components of G is equal to *one*, then G is, of course, connected.

The unit graph $G(\mathbb{Z}_2 \times \mathbb{Z}_2)$, shown in Fig. 1, is not connected. There are also many other rings which have disconnected unit graphs. For example, suppose that

$R = S \times T$ where $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and T is a ring. Then clearly $G(R)$ is a union of two copies of vertex-disjoint graph $G(\mathbb{Z}_2 \times T)$, and therefore, $G(R)$ is not connected. Thus unit graphs are not connected in general. In this section we study when unit graphs are connected. First, we give a necessary and sufficient condition for connectedness of $G(R)$ based on the unit sum number of R. In order to do this, we first explain a few things about k-good rings.

Definitions and Remarks 4.2. Let R be a ring and k be a positive integer. An element $r \in R$ is said to be *k-good* if we may write $r = u_1 + \cdots + u_k$ where $u_1, \ldots, u_k \in U(R)$. The ring R is said to be *k-good* if every element of R is k-good. Following [6] we now define an invariant of a ring, called the *unit sum number*, which expresses, in a fairly precise way, how the units generate the ring. The *unit sum number* of R, denoted by $\mathbf{u}(R)$, is given by:

a) $\min\{k \mid R \text{ is } k\text{-good}\}$ if R is k-good for some $k \geq 1$;
b) ω if R is not k-good for every k, but every element of R is k-good for some k (that is, when at least $U(R)$ generates R additively);
c) ∞ otherwise (that is, when $U(R)$ does not generate R additively).

For example, let D be a division ring. If $|D| \geq 3$, then $\mathbf{u}(D) = 2$; whereas if $|D| = 2$, that is, $D = \mathbb{Z}_2$ the field of two elements, then $\mathbf{u}(\mathbb{Z}_2) = \omega$. We have also $\mathbf{u}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \infty$ (cf. [1] for unit sum numbers of some other rings).

The topic of unit sum numbers seems to have arisen in 1954 with an article by Zelinsky [18] in which he shows that if V is any finite or infinite dimensional vector space over a division ring D then every linear transformation is the sum of two automorphisms unless $\dim V = 1$ and D is the field of two elements. Interest in this topic increased recently after Goldsmith, Pabst and Scott defined the unit sum number in [6]. For additional historical background the reader is referred to the article [17], which also contains references to recent work in this area.

We now state Theorem 4.3, which contains a necessary and sufficient condition for $G(R)$ to be connected, based on the unit sum number of R.

Theorem 4.3. *Let R be a ring. Then the unit graph $G(R)$ is connected if and only if $\mathbf{u}(R) \leq \omega$.*

Proof. Consider

$$S = \{a \in R \mid a \text{ is } k\text{-good for some } k\}.$$

Note that 0 is a 2-good element of R and so $0 \in S$. (In fact, S is a subring of R.) We claim that the vertex-induced subgraph $\langle S \rangle$ of $G(R)$ is a connected component of $G(R)$. For proving the claim suppose that $x \in S$ is given. Therefore x is a k-good element for some k, and so we may write $x = u_1 + \cdots + u_k$ where $u_1, \ldots, u_k \in U(R)$. We now have the walk

$$0 \xrightarrow{e_1} -u_1 \xrightarrow{e_2} u_1 + u_2 \xrightarrow{e_3} -u_1 - u_2 - u_3 \xrightarrow{e_4} u_1 + u_2 + u_3 + u_4 \longrightarrow \ldots$$
$$\xrightarrow{e_k} u_1 + \cdots + u_k = x$$

between 0 and x in $\langle S\rangle$, when k is even, and the walk

$$0 \xrightarrow{e_1} u_1 \xrightarrow{e_2} -u_1 - u_2 \xrightarrow{e_3} u_1 + u_2 + u_3 \xrightarrow{e_4} -u_1 - u_2 - u_3 - u_4 \longrightarrow \dots \xrightarrow{e_k} u_1 + \cdots + u_k = x$$

between 0 and x in $\langle S\rangle$, when k is odd.

This implies that for every $x, y \in S$, there is a walk W_1 between x and 0 in $\langle S\rangle$ as well as a walk W_2 between 0 and y in $\langle S\rangle$. The walks W_1 and W_2 together form a walk W between x and y in $\langle S\rangle$. Thus, we conclude that there is also a path P between x and y in $\langle S\rangle$. Therefore, $\langle S\rangle$ is a connected subgraph of $G(R)$.

Now suppose that $x \in R\backslash S$. If the vertex-induced subgraph $\langle S \cup \{x\}\rangle$ of $G(R)$ is still connected, then there exists a path

$$0 = x_0 \xrightarrow{e_1} x_1 \xrightarrow{e_2} x_2 \longrightarrow \cdots \xrightarrow{e_k} x_k \xrightarrow{e_{k+1}} x_{k+1} = x$$

between 0 and x in $\langle S \cup \{x\}\rangle$. Thus $x_i + x_{i-1} := u_i$ is a unit for $1 \le i \le k+1$ and so

$$x = \begin{cases} \sum_{i=0}^{k} (-1)^i u_{i+1} & \text{if } k \text{ is even,} \\ \sum_{i=0}^{k} (-1)^{i+1} u_{i+1} & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, $x \in S$, a contradiction. This implies that $\langle S\rangle$ is a connected component of $G(R)$.

Now it is easy to see that $\mathbf{u}(R) \le \omega$ if and only if $S = R$. On the other hand, $S = R$ if and only if $\langle S\rangle = G(R)$. Therefore, using the claim, we conclude that $G(R)$ is connected if and only if $\mathbf{u}(R) \le \omega$. □

Note that there are infinitely many rings R such that $\mathbf{u}(R) \le \omega$ and, therefore, by Theorem 4.3, their unit graphs are connected (see for example [8, 16]).

We recall that for a graph G, a subset S of the vertex-set of G is called a *dominating set* if every vertex not in S is adjacent to a vertex in S.

Corollary 4.4. *Let R be a ring. If $U(R)$ is a dominating set of $G(R)$, then the unit graph $G(R)$ is connected.*

Proof. Suppose that $x \in R\backslash U(R)$ is given. Since $U(R)$ is a dominating set, there exists $u_1 \in U(R)$ such that x is adjacent to u_1. Therefore, $x + u_1 \in U(R)$, and so x is a 2-good element. This implies that $\mathbf{u}(R) \le \omega$ and so by Theorem 4.3, $G(R)$ is connected as required. □

Note that there are infinitely many rings R such that $U(R)$ is a dominating set of $G(R)$ and, therefore, by Corollary 4.4, the unit graph $G(R)$ is connected. For example, if R is a 2-good ring, then $U(R)$ is a dominating set of $G(R)$. To show this fact, let $x \in R\backslash U(R)$ be given. We may write $x = u_1 + u_2$, where $u_1, u_2 \in U(R)$. This implies that x is adjacent to $-u_1 \in U(R)$ and, therefore, $U(R)$ is a dominating set. It is remarkable that the converse of this fact is not correct in general. For example $U(\mathbb{Z}_2) = \{1\}$ is a dominating set of $G(\mathbb{Z}_2)$, but $\mathbb{Z}_2$ is not a 2-good ring since $\mathbf{u}(\mathbb{Z}_2) = \omega$.

In the following, we study the connectedness of the unit graphs associated with certain particular types of rings. We start with rings of polynomials. For the proof of the next result, we need a well-known fact, and so we state it here for the convenience of the reader.

Fact 4.5 (cf. [9]). Let R be a commutative ring and $f = a_0 + a_1x + \cdots + a_nx^n \in R[x]$. Then f is a unit in $R[x]$ if and only if a_0 is a unit in R and for every i, $1 \leq i \leq n$, a_i is a nilpotent element of R.

Proposition 4.6. *Let R be a commutative ring. If R is reduced then the unit graph $G(R[x])$ is a disconnected graph.*

Proof. By Fact 4.5, if two polynomials $f = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ and $g = b_0 + b_1x + \cdots + b_mx^m \in R[x]$ are adjacent, then necessarily each $a_i = -b_i$ whenever $i > 0$ and so $n = m$. Therefore, if there exists a path between $g_1, g_2 \in R[x]$, then $\deg(g_1) = \deg(g_2)$. This means that $G(R[x])$ is disconnected. □

The following result gives us information on the connectedness of unit graphs associated with matrix rings. Note that this result could also be deduced from Theorem 4.3, since Henriksen [8] showed that the unit sum number of the ring of $n \times n$ matrices over R, $M_n(R)$, $n > 1$, is less than or equal to three.

Proposition 4.7. *Let R be a field with $|R| > 2$, and let $M_n(R)$, $n > 1$, be the ring of $n \times n$ matrices over R. Then the unit graph $G(M_n(R))$ is a connected graph.*

Proof. Suppose that A and B are two given elements of $M_n(R)$. By a result due to I. Kaplansky (cf. [8, Lemma 2]), we may write $A = D_A + U_A$ and $B = D_B + U_B$, where D_A and D_B are diagonal matrices in $M_n(R)$ and U_A and U_B are unit elements of $M_n(R)$. Therefore, A is adjacent to $-D_A$ and B is adjacent to $-D_B$. Consider $-D_A = \mathrm{diag}(d_1, \ldots, d_n)$ and $-D_B = \mathrm{diag}(d'_1, \ldots, d'_n)$. Since $|R| > 2$, for each i, $1 \leq i \leq n$, we may choose l_i such that $l_i \notin \{-d_i, -d'_i\}$. If we now consider $L = \mathrm{diag}(l_1, \ldots, l_n)$, then both of $-D_A$ and $-D_B$ are adjacent to L. This shows that the unit graph $G(M_n(R))$ is connected as required. □

By applying Maschke's theorem and Wedderburn–Artin theorem, the above proposition implies the following result.

Corollary 4.8. *Let R be an artinian semisimple ring, G be a finite group such that $|G|$ is a unit in R and let RG be the group ring. Then the unit graph $G(RG)$ is a connected graph.*

5. FURTHER PROPERTIES OF UNIT GRAPHS

In this section some characterization results regarding chromatic index, diameter, girth, and planarity of $G(R)$ are given.

Definitions and Remarks 5.1. A *k-edge coloring* of a graph G is an assignment of k *labels*, also called *colors*, to the edges of G such that every pair of distinct edges meeting at a common vertex are assigned two different colors. If G has a k-edge coloring, then G is said to be *k-edge colorable*. The *chromatic index* of G, denoted $\chi'(G)$, is the smallest number k such that G is k-edge colorable. By Vizing's theorem, if G is a graph whose maximum vertex degree is Δ, then $\Delta \leq \chi'(G) \leq \Delta + 1$. Vizing's theorem divides the graphs into two classes according to their chromatic index; graphs satisfying $\chi'(G) = \Delta$ are called *class* 1, those with $\chi'(G) = \Delta + 1$ are *class* 2.

We now state the following result which shows that all unit graphs are class 1.

Theorem 5.2. *Let R be a finite ring. Then the unit graph $G(R)$ is class* 1.

Proof. We color the edge xy by the color $x + y$. By this coloring, every two distinct edges xy and xz have different colors and $C = \{x + y \mid xy \text{ is an edge of } G\}$ is the set of colors. Therefore, $G(R)$ has a $|C|$-edge coloring and so $\chi'(G(R)) \leq |C|$. But clearly $C \subseteq U(R)$ and thus $\chi'(G(R)) \leq |C| \leq |U(R)|$. On the other hand, Proposition 2.4 implies that $\Delta = |U(R)|$ and so by Vizing's theorem we have $\chi'(G(R)) \geq \Delta = |U(R)|$. Therefore $\chi'(G(R)) = |U(R)| = \Delta$ and so $G(R)$ is class 1. □

Definitions and Remarks 5.3. For a graph G and vertices x and y of G, the *distance* between x and y, denoted $d(x, y)$, is the number of edges in a shortest path between x and y. If there is no any path between x and y, then we write $d(x, y) = \infty$. For example, let R be a ring, $G(R)$ be the unit graph associated with R and $x \in R$. Then the proof of Theorem 4.3 shows that $d(0, x) = k$ if and only if x is a k-good element of R but it is not a $(k - 1)$-good element. Also we recall that the largest distance among all distances between pairs of the vertices of a graph G is called the *diameter* of G and is denoted by $\operatorname{diam}(G)$. One can easily see that, for example, $\operatorname{diam}(G(\mathbb{Z}_6)) = 3$, $\operatorname{diam}(G(\mathbb{Z}_8)) = 2$, and $\operatorname{diam}(G(\mathbb{Z})) = \infty$.

The following result is an immediate consequence of the proof of Proposition 4.7.

Corollary 5.4. *Let R be a field with $|R| > 2$, and let $M_n(R)$, $n > 1$, be the ring of $n \times n$ matrices over R. Then we have* $\operatorname{diam}(G(M_n(R))) \leq 4$.

In the following, we study the diameter of the unit graphs. By using Lemma 5.5 and Corollary 5.6, we prove Theorem 5.7 which is a characterization of diameter of the unit graph $G(R)$.

Lemma 5.5. *Let R be a finite commutative ring such that R is not a field with* $\operatorname{Char}(R) = 2$. *Then the following statements hold*:

(a) *If either R cannot have $\mathbb{Z}_2$ as a quotient or R is a local ring with $\mathfrak{m}$ as the unique maximal ideal in such a way that $|R/\mathfrak{m}| = 2$, then* $\operatorname{diam}(G(R)) = 2$;
(b) *If R has $\mathbb{Z}_2$ as a quotient and R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient and R is not a local ring with $\mathfrak{m}$ as the unique maximal ideal in such a way that $|R/\mathfrak{m}| = 2$, then* $\operatorname{diam}(G(R)) = 3$;
(c) *If R has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, then* $\operatorname{diam}(G(R)) = \infty$.

Proof. Since every finite commutative ring with nonzero identity is isomorphic to a direct product of finite local rings (cf. [13, p. 95]), we may write $R \cong R_1 \times \cdots \times R_k$, where every R_i is a local ring with maximal ideal $\mathfrak{m}_i$.

Now for proof of (a), assume that R cannot have $\mathbb{Z}_2$ as a quotient. Thus $|R_i/\mathfrak{m}_i| > 2$ holds for every i. Suppose $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ are arbitrary distinct elements of $R_1 \times \cdots \times R_k$. Since for every i with $1 \leq i \leq k$, we have $|R_i/\mathfrak{m}_i| > 2$, by Remark 2.5, we conclude that $N_{\overline{G}(R_i)}(x_i) \cap N_{\overline{G}(R_i)}(y_i) \neq \emptyset$. Therefore, we may choose $z_i \in N_{\overline{G}(R_i)}(x_i) \cap N_{\overline{G}(R_i)}(y_i)$. Thus we have the following walk in $\overline{G}(R_1 \times \cdots \times R_k)$:

$$(x_1, \ldots, x_k) \xrightarrow{e_1} (z_1, \ldots, z_k) \xrightarrow{e_2} (y_1, \ldots, y_k).$$

This implies that $d(x, y) \leq 2$ and so $\operatorname{diam}(G(R)) = \operatorname{diam}(G(R_1 \times \cdots \times R_k)) \leq 2$. If $\operatorname{diam}(G(R)) = 1$, then $G(R)$ is a complete graph and so by Theorem 3.4, R is a field with $\operatorname{Char}(R) = 2$, a contradiction. Therefore, $\operatorname{diam}(G(R)) = 2$.

Now suppose that R is a finite local ring with maximal ideal $\mathfrak{m}$ in such a way that $|R/\mathfrak{m}| = 2$. By using Theorem 3.5 we conclude that $G(R)$ is complete bipartite with $|R| \geq 4$, and so $\operatorname{diam}(G(R)) = 2$.

For proof of (b), since R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient but has $\mathbb{Z}_2$ as a quotient, $|R_i/\mathfrak{m}_i| > 2$ holds for every i except one of them. If $k = 1$, then $R \cong R_1$ is a finite local ring with maximal ideal $\mathfrak{m}_1$ in such a way that $|R_1/\mathfrak{m}_1| = 2$, a contradiction. Therefore, $k \geq 2$. In this case, we may assume that $|R_1/\mathfrak{m}_1| = 2$ and $|R_i/\mathfrak{m}_i| > 2$ for every i with $2 \leq i \leq k$. Suppose that $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ are arbitrary distinct elements of $R_1 \times R_2 \times \cdots \times R_k$. If either $x_1, y_1 \in \mathfrak{m}_1$ or $x_1, y_1 \notin \mathfrak{m}_1$, then by an argument similar to that above, we obtain a path between x and y with the length at most 2. This implies that $d(x, y) \leq 2$. Now, we may assume that $x_1 \in \mathfrak{m}_1$ and $y_1 \notin \mathfrak{m}_1$. For every i with $2 \leq i \leq k$, consider w_i as follows:

$$w_i = \begin{cases} 1 & \text{if } x_i \in \mathfrak{m}_i, \\ 0 & \text{if } x_i \notin \mathfrak{m}_i. \end{cases}$$

On the other hand, since for every i with $2 \leq i \leq k$ we have $|R_i/\mathfrak{m}_i| > 2$, by Remark 2.5, we conclude that $N_{\overline{G}(R_i)}(w_i) \cap N_{\overline{G}(R_i)}(y_i) \neq \emptyset$. Therefore, we may choose $z_i \in N_{\overline{G}(R_i)}(w_i) \cap N_{\overline{G}(R_i)}(y_i)$. Thus we have the following walk in $\overline{G}(R_1 \times R_2 \times \cdots \times R_k)$:

$$(x_1, x_2, \ldots, x_k) \xrightarrow{e_1} (y_1, w_2, \ldots, w_k) \xrightarrow{e_2} (x_1, z_2, \ldots, z_k) \xrightarrow{e_3} (y_1, y_2, \ldots, y_k).$$

This implies that $d(x, y) \leq 3$. Therefore, for every distinct elements $x, y \in R_1 \times R_2 \times \cdots \times R_k$, we have $d(x, y) \leq 3$ and so $\operatorname{diam}(G(R_1 \times R_2 \times \cdots \times R_k)) \leq 3$. Now, consider two elements $a = (0, 0, \ldots, 0)$ and $b = (1, 0, \ldots, 0)$ in $G(R_1 \times R_2 \times \cdots \times R_k)$. Since a and b are not adjacent, we conclude that $d(a, b) \geq 2$. If $d(a, b) = 2$, then there must be a path

$$(0, 0, \ldots, 0) \xrightarrow{e_1} (x_1, x_2, \ldots, x_k) \xrightarrow{e_2} (1, 0, \ldots, 0).$$

The adjacency of $(0, 0, \ldots, 0)$ and $(x_1, x_2, \ldots, x_k)$ implies that x_1 is a unit element, while the adjacency of $(x_1, x_2, \ldots, x_k)$ and $(1, 0, \ldots, 0)$ implies that x_1 is not a unit element, a contradiction. Therefore, $d(a, b) \geq 3$ and since $\operatorname{diam}(G(R_1 \times R_2 \times \cdots \times R_k)) \leq 3$, we conclude that $\operatorname{diam}(G(R)) = \operatorname{diam}(G(R_1 \times R_2 \times \cdots \times R_k)) = 3$.

For (c), we have $k \geq 2$, and so we may consider two elements $a = (1, 1, \ldots, 0)$ and $b = (1, 0, \ldots, 0)$ in $G(R_1 \times R_2 \times \cdots \times R_k)$. If there is a path

$$(1, 1, \ldots, 0) \xrightarrow{e_1} (x_1, x_2, \ldots, x_k) \longrightarrow \cdots \xrightarrow{e_{n-1}} (y_1, y_2, \ldots, y_k) \xrightarrow{e_n} (1, 0, \ldots, 0)$$

between a and b, then we obtain the following walks in $G(R_1)$ and $G(R_2)$:

$$1 \xrightarrow{e_1'} x_1 \longrightarrow \cdots \xrightarrow{e_{n-1}'} y_1 \xrightarrow{e_n'} 1,$$

$$1 \xrightarrow{e_1''} x_2 \longrightarrow \cdots \xrightarrow{e_{n-1}''} y_2 \xrightarrow{e_n''} 0.$$

The first walk shows that n is even, while the second one shows that n is odd, a contradiction. Therefore, there is no path between a and b which implies that $\operatorname{diam}(G(R)) = \operatorname{diam}(G(R_1 \times R_2 \times \cdots \times R_k)) = \infty$. □

The following result shows that the diameter of the unit graph of a finite commutative ring does not contain all positive integers.

Corollary 5.6. *Let R be a finite commutative ring. Then we have* $\operatorname{diam}(G(R)) \in \{1, 2, 3, \infty\}$.

Proof. If R is a field with $\operatorname{Char}(R) = 2$, then by Theorem 3.4, $G(R)$ is a complete graph and so $\operatorname{diam}(G(R)) = 1$. If R is a field with $\operatorname{Char}(R) \neq 2$, then by Theorem 3.8, $\operatorname{diam}(G(R)) = 2$.

Now, we assume that R is not a field. If R has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, then part (c) of Lemma 5.5 implies that $\operatorname{diam}(G(R)) = \infty$. If R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, then there are two possibilities: either R cannot have $\mathbb{Z}_2$ as a quotient or has $\mathbb{Z}_2$ as a quotient. In the first case, by part (a) of Lemma 5.5, we have $\operatorname{diam}(G(R)) = 2$. In the second case, if R is a local ring with maximal ideal $\mathfrak{m}$, then $|R/\mathfrak{m}| = 2$ and so by part (a) of Lemma 5.5, we have $\operatorname{diam}(G(R)) = 2$. Otherwise, by part (b) of Lemma 5.5, we have $\operatorname{diam}(G(R)) = 3$. □

The following theorem is a characterization of diameter of the unit graph $G(R)$.

Theorem 5.7. *Let R be a finite commutative ring. Then the following statements hold*:

(a) $\operatorname{diam}(G(R)) = 1$ *if and only if R is a field with* $\operatorname{Char}(R) = 2$;
(b) $\operatorname{diam}(G(R)) = 2$ *if and only if one of the following cases occurs*:

  (1) *R is a field with* $\operatorname{Char}(R) \neq 2$;
  (2) *R is not a field and R cannot have $\mathbb{Z}_2$ as a quotient*;
  (3) *R is a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$ and $R \ncong \mathbb{Z}_2$*;

(c) $\operatorname{diam}(G(R)) = 3$ *if and only if R has* $\mathbb{Z}_2$ *as a quotient and cannot have* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *as a quotient and R is not a local ring*;
(d) $\operatorname{diam}(G(R)) = \infty$ *if and only if R has* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *as a quotient.*

Proof. Part (a) is an immediate consequence of Theorem 3.4.

For (b), first assume that $\operatorname{diam}(G(R)) = 2$. There are two possibilities: either R is a filed or not. If R is a field, then by part (a), we conclude that $\operatorname{Char}(R) \neq 2$. Thus the case (1) occurs. Therefore, we may assume that R is not a field. If R cannot have $\mathbb{Z}_2$ as a quotient then, of course, the case (2) occurs. Thus we may suppose that R has $\mathbb{Z}_2$ as a quotient. By part (c) of Lemma 5.5, R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient and so part (b) of Lemma 5.5 implies that R is a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$ and $R \ncong \mathbb{Z}_2$. Thus the case (3) occurs.

The converse of part (b) is an immediate consequence of Theorem 3.8 and Lemma 5.5.

For (c), first assume that $\operatorname{diam}(G(R)) = 3$. Note that the proof of Corollary 5.6 implies that R is not a field. Therefore, by part (c) of Lemma 5.5, R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. If R cannot have $\mathbb{Z}_2$ as a quotient, then by part (a) of Lemma 5.5, $\operatorname{diam}(G(R)) = 2$, a contradiction. Thus R has $\mathbb{Z}_2$ as a quotient. If R is a local ring with maximal ideal $\mathfrak{m}$, then $|R/\mathfrak{m}| = 2$. Thus by part (a) of Lemma 5.5, $\operatorname{diam}(G(R)) = 2$, a contradiction. This completes the proof.

Part (d) and the converse of part (c) is an immediate consequence of Lemma 5.5 and Corollary 5.6. □

Remark 5.8. The finiteness of R in Theorem 5.7 is essential. For example, the unit graphs $G(\mathbb{Z})$ and $G(\mathbb{Z} \times \mathbb{Z})$ are both connected, and we have $\operatorname{diam}(G(\mathbb{Z})) = \operatorname{diam}(G(\mathbb{Z} \times \mathbb{Z})) = \infty$. Note that $\mathbb{Z}$ has $\mathbb{Z}_2$ as a quotient and cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, while $\mathbb{Z} \times \mathbb{Z}$ has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.

Definitions and Remarks 5.9. For a graph G, the *girth* of G is the length of a shortest cycle in G and is denoted by $\operatorname{gr}(G)$. If G has no cycles, we define the girth of G to be infinite. If G is a bipartite graph, then G has no odd cycle and so $\operatorname{gr}(G) \geq 4$ (cf. Definitions and Remarks 3.1).

The following result shows that the girth of the unit graph of a finite commutative ring does not contain all positive integers.

Proposition 5.10. *Let R be a finite commutative ring. Then we have* $\operatorname{gr}(G(R)) \in \{3, 4, 6, \infty\}$.

Proof. Suppose that $J(R) \neq 0$, where $J(R)$ denotes the Jacobson radical of R. Therefore we may choose two distinct elements x and y in $J(R)$. This implies that two distinct elements $1 - x$ and $1 - y$ lie in $U(R)$. On the other hand, every element of $J(R)$ is adjacent to every element of $U(R)$. Hence we obtain that $\operatorname{gr}(G(R)) \in \{3, 4\}$.

Now suppose that $J(R) = 0$. Since every finite commutative ring with nonzero identity is isomorphic to a direct product of finite local rings (cf. [13, p. 95]), we may write $R \cong R_1 \times \cdots \times R_k$, where every R_i is a local ring with maximal ideal $\mathfrak{m}_i$. If $k = 1$, then R is a field and so $R \cong \mathbb{Z}_2$, $R \cong \mathbb{Z}_3$ or $|R| \geq 4$. If $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$, then we have $\mathrm{gr}(G(R)) = \infty$. If $|R| \geq 4$, then we have two nonzero unit elements x and y such that $x + y \neq 0$. Therefore, x and y are adjacent, and so we obtain the cycle

$$0 \xrightarrow{e_1} x \xrightarrow{e_2} y \xrightarrow{e_3} 0.$$

This implies that $\mathrm{gr}(G(R)) = 3$. If $k \geq 2$, then for every i with $1 \leq i \leq k$ we have $J(R_i) = 0$ and so R_i is a field. Therefore, we may assume that R has the form

$$R \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell \text{ times}} \times \underbrace{\mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3}_{m \text{ times}} \times F_1 \times \cdots \times F_t \times E_1 \times \cdots \times E_s,$$

where $\mathrm{Char}(F_i) = 2$, $|F_i| \geq 4$, $1 \leq i \leq t$ and $\mathrm{Char}(E_j) \geq 3$, $|E_i| \geq 4$, $1 \leq j \leq s$, and $\ell + m + t + s = k$. We consider the following cases.

Case 1: If $\ell = k$. In this case, R has just one unit element and so $G(R)$ is a 1-regular graph. Thus $\mathrm{gr}(G(R)) = \infty$.

Case 2: If $0 < \ell < k$ and $m = 0$. In this case, $G(R)$ is a union of some copies of

$$G(\mathbb{Z}_2 \times F_1 \times \cdots \times F_t \times E_1 \times \cdots \times E_s),$$

where every copy is a bipartite graph. Thus $\mathrm{gr}(G(R)) \geq 4$. Now, for every i with $1 \leq i \leq t$ and j with $1 \leq j \leq s$ we choose the nonzero elements $\alpha_i, \beta_i, \gamma_i \in F_i$ and $\delta_j, \eta_j \in E_j$ in such a way that $\alpha_i + \beta_i \neq 0$, $\beta_i + \gamma_i \neq 0$, and $\delta_j + \eta_j \neq 0$. Therefore, we obtain the cycle

$$\begin{aligned}(0, \alpha_1, \ldots, \alpha_t, 0, \ldots, 0) &\xrightarrow{e_1} (1, \beta_1, \ldots, \beta_t, \eta_1, \ldots, \eta_s) \xrightarrow{e_2} (0, \gamma_1, \ldots, \gamma_t, 0, \ldots, 0) \\ &\xrightarrow{e_3} (1, 0, \ldots, 0, \delta_1, \ldots, \delta_s) \xrightarrow{e_4} (0, \alpha_1, \ldots, \alpha_t, 0, \ldots, 0),\end{aligned}$$

which implies that $\mathrm{gr}(G(R)) = 4$.

Case 3: If $0 < \ell < k$ and $m \neq 0$. By an argument similar to case 2, we consider the unit graph

$$G(\mathbb{Z}_2 \times \underbrace{\mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3}_{m \text{ times}} \times F_1 \times \cdots \times F_t \times E_1 \times \cdots \times E_s)$$

and obtain the cycle

$$\begin{aligned}&(0, 0, \ldots, 0, \alpha_1, \ldots, \alpha_t, 0, \ldots, 0) \\ &\xrightarrow{e_1} (1, 2, \ldots, 2, 0, \ldots, 0, \eta_1, \ldots, \eta_s) \xrightarrow{e_2} (0, 2, \ldots, 2, \beta_1, \ldots, \beta_t, 0, \ldots, 0) \\ &\xrightarrow{e_3} (1, 0, \ldots, 0, 0, \ldots, 0, \delta_1, \ldots, \delta_s) \xrightarrow{e_4} (0, 1, \ldots, 1, \gamma_1, \ldots, \gamma_t, 0, \ldots, 0) \\ &\xrightarrow{e_5} (1, 1, \ldots, 1, 0, \ldots, 0, \eta_1, \ldots, \eta_s) \xrightarrow{e_6} (0, 0, \ldots, 0, \alpha_1, \ldots, \alpha_t, 0, \ldots, 0).\end{aligned}$$

This implies that $\mathrm{gr}(G(R)) \leq 6$. But in this case, the unit graph $G(R)$ is bipartite, and so we get $\mathrm{gr}(G(R)) \in \{4, 6\}$.

Case 4: If $\ell = 0$. By an argument similar to case 2, we obtain the cycle

$$\begin{aligned}(0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0) &\xrightarrow{e_1} (1, \ldots, 1, \alpha_1, \ldots, \alpha_t, \delta_1, \ldots, \delta_s)\\ &\xrightarrow{e_2} (1, \ldots, 1, \beta_1, \ldots, \beta_t, \eta_1, \ldots, \eta_s)\\ &\xrightarrow{e_3} (0, \ldots, 0, 0, \ldots, 0, 0, \ldots, 0).\end{aligned}$$

Thus $\mathrm{gr}(G(R)) = 3$. This completes the proof. □

Remark 5.11. Note that $\mathrm{gr}(G(\mathbb{Z}_5)) = 3$, $\mathrm{gr}(G(\mathbb{Z}_4)) = 4$, $\mathrm{gr}(G(\mathbb{Z}_6)) = 6$, and $\mathrm{gr}(G(\mathbb{Z}_2)) = \infty$. Therefore, for every $n \in \{3, 4, 6, \infty\}$, there exists a ring R such that $\mathrm{gr}(G(R)) = n$.

The following result is an immediate consequence of the proof of Proposition 2.9.

Corollary 5.12. *Let R be a finite commutative ring, and let M be a finite unitary R-module. Suppose that $R \oplus M$ denotes the Nagata extension of R by M. If $|M| > 2$, then we have* $\mathrm{gr}(G(R \oplus M)) \leq 4$. *In particular, if* $2 \in U(R)$, *then* $\mathrm{gr}(G(R \oplus M)) = 3$.

Definitions and Remarks 5.13. A graph G is said to be *planar* if it can be drawn in the plane in such a way that its edges intersect only at their ends. A *subdivision of an edge* of G is obtained by inserting into this edge some new vertices of degree two. A *subdivision of a graph* G is the result of subdividing its edges. Further, G is also considered as a subdivision of itself. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's theorem says that a graph is planar if and only if it contains no subdivision of $K_{3,3}$ or K_5 (cf. Definitions and Remarks 2.6 and [3, p. 153]). Also if G is a connected planar graph with $p \geq 3$ vertices and q edges, then we have $q \leq 3p - 6$. Moreover, if $\mathrm{gr}(G) \geq 4$, then we have the better bound $q \leq 2p - 4$. In addition, if G is a planar graph, then G contains a vertex of degree at most five. Unit graphs are not planar in general. An example to illustrate this fact is the unit graph $G(\mathbb{Z}_{10})$, which is not planar because $G(\mathbb{Z}_{10})$ is a 4-regular connected bipartite graph and thus has no odd cycle (cf. Definitions and Remarks 3.1). Therefore $\mathrm{gr}(G(\mathbb{Z}_{10})) \geq 4$ (cf. Definitions and Remarks 5.9) and so $q = 20 > 16 = 2p - 4$ which, by the above observation, implies that $G(\mathbb{Z}_{10})$ is not planar. On the other hand, there are also planar unit graphs. For example, the unit graph $G(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a planar graph (cf. Fig. 4).

We now state our final result: a necessary and sufficient condition for the planarity of $G(R)$. Here, $\mathbb{F}_4$ denotes the field with four elements.

Theorem 5.14. *Let R be a finite commutative ring. Then the unit graph $G(R)$ is planar if and only if R is isomorphic to one of the following rings*:

(a) $\mathbb{Z}_5$;

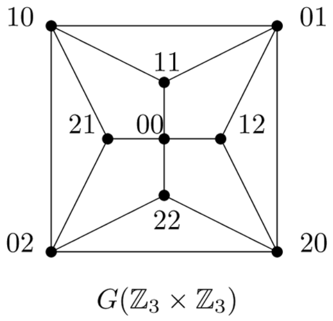

Figure 4 A planar graph unit.

(b) $\mathbb{Z}_3 \times \mathbb{Z}_3$;
(c) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell \text{ times}} \times S$, *where* $\ell \geq 0$ *and* $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$, $S \cong \mathbb{Z}_4$, $S \cong \mathbb{F}_4$, *or* $S \cong \left\{ \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2 \right\}$.

Proof. Suppose that the unit graph $G(R)$ is planar. We claim that $|J(R)| \leq 2$, where $J(R)$ denotes the Jacobson radical of R. To prove this, we suppose the contrary. Then $|J(R)| \geq 3$. Therefore we may choose three distinct elements x, y, and z in $J(R)$. This implies that three distinct elements $1 - x$, $1 - y$, and $1 - z$ lie in $U(R)$. Since every element of $J(R)$ is adjacent to every element of $U(R)$, we obtain a graph $K_{3,3}$ in the structure of $G(R)$. Thus, by Kuratowski's theorem (cf. Definitions and Remarks 5.13), the unit graph $G(R)$ is not planar, a contradiction. Therefore, the claim holds, and we have $|J(R)| \leq 2$.

On the other hand, by the observation in Definitions and Remarks 5.13, the minimum degree of vertices of $G(R)$ is at most five. Thus Proposition 2.4 implies that $|U(R)| \leq 6$.

Since every finite commutative ring with nonzero identity is isomorphic to a direct product of finite local rings (cf. [13, p. 95]), we may write $R \cong R_1 \times \cdots \times R_k$, where every R_i is a local ring with maximal ideal $\mathfrak{m}_i$. There are two cases to be considered: (1) $k = 1$ and (2) $k \geq 2$.

Case 1: In this case, we have $k = 1$ and so R ($\cong R_1$) is a local ring with maximal ideal $\mathfrak{m}$ ($\cong \mathfrak{m}_1$). On the other hand, by the above observation, we have $|J(R)| \leq 2$ and $|U(R)| \leq 6$. Thus we obtain that $|\mathfrak{m}| \leq 2$ and $|R \backslash \mathfrak{m}| \leq 6$.

If $|\mathfrak{m}| = 1$, then $\mathfrak{m} = \{0\}$ and so R is a field. Since $|R \backslash \mathfrak{m}| \leq 6$, we have that $|R| \leq 7$ and so $R \cong \mathbb{Z}_2$, $R \cong \mathbb{Z}_3$, $R \cong \mathbb{F}_4$, $R \cong \mathbb{Z}_5$, or $R \cong \mathbb{Z}_7$. The case $R \cong \mathbb{Z}_7$ is ruled out since the unit graph $G(\mathbb{Z}_7)$ is not planar (cf. Definitions and Remarks 5.13). The other rings have planar unit graphs.

If $|\mathfrak{m}| = 2$, then $\mathfrak{m} \neq \{0\}$, and so R is not a field. Since $|R \backslash \mathfrak{m}| \leq 6$, we have that $|R| \leq 8$ and so $|R| = 4$ or $|R| = 8$. If $|R| = 4$, then $R \cong \mathbb{Z}_4$ or $R \cong \left\{ \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2 \right\}$, which both have planar unit graphs. If $|R| = 8$, then R is isomorphic to one of the five local rings of order 8: $R \cong L_i$, $1 \leq i \leq 5$ (cf. [4]). The unit graphs associated with all of these rings are ruled out because for every i with $1 \leq i \leq 5$, we have $|J(L_i)| = 4$, and so the associated unit graphs are not planar.

Case 2: In this case, we have $k \geq 2$.

If $|J(R)| = 1$, then for every i with $1 \leq i \leq k$, we have $|J(R_i)| = 1$ and so R_i is a field. On the other hand, $|U(R)| \leq 6$, and so for every i with $1 \leq i \leq k$, we have $|U(R_i)| \leq 6$. Therefore, R is isomorphic to one of the following rings:

(i) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell \text{ times}} \times S$, where $\ell > 0$ and $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$, $S \cong \mathbb{F}_4$, $S \cong \mathbb{Z}_5$, or $S \cong \mathbb{Z}_7$;

(ii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell \text{ times}} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, where $\ell \geq 0$;

(iii) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell \text{ times}} \times \mathbb{Z}_3 \times \mathbb{F}_4$, where $\ell \geq 0$.

If R is isomorphic to one the rings that appear in (i), then clearly $G(R)$ is a union of some copies of vertex-disjoint graph $G(\mathbb{Z}_2 \times S)$. Therefore, the planarity of $G(R)$ is equivalent to planarity of $G(\mathbb{Z}_2 \times S)$. On the other hand, both $G(\mathbb{Z}_2 \times \mathbb{Z}_5)$ and $G(\mathbb{Z}_2 \times \mathbb{Z}_7)$ are bipartite graphs with girth ≥ 4 and so, by Definitions and Remarks 5.13, are not planar. Thus the cases $S \cong \mathbb{Z}_5$ and $S \cong \mathbb{Z}_7$ among the rings appearing in (i) are ruled out. The other rings in (i) have planar unit graphs.

For rings appearing in (ii), if $\ell = 0$, then we have the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$ which has planar unit graph (cf. Fig. 4). If $\ell > 0$, then all rings are ruled out. To see this, we note that since the unit graph $G(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ is 4-regular with girth ≥ 4 and 36 edges, Definitions and Remarks 5.13 imply it is not planar.

All rings appearing in (iii) are ruled out. To see this, we note that the unit graph $G(\mathbb{Z}_3 \times \mathbb{F}_4)$ is 6-regular, and so the minimum degree of vertices of $G(\mathbb{Z}_3 \times \mathbb{F}_4)$ is six. Therefore, by Definitions and Remarks 5.13, the unit graph $G(\mathbb{Z}_3 \times \mathbb{F}_4)$ is not planar.

If $|J(R)| = 2$, then for at most one i with $1 \leq i \leq k$, we have $|J(R_i)| = 2$. Thus $R \cong R_1 \times \cdots \times R_{k-1} \times S$, where every R_i is a field and $|J(S)| = 2$. Since $|U(R)| \leq 6$, we conclude that $|U(S)| \leq 6$. Therefore, $|S| \leq 8$ and so $S \cong \mathbb{Z}_4$ or $S \cong \left\{ \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2 \right\}$. By an argument similar to that above, we conclude that if

$$R \cong \underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell \text{ times}} \times S,$$

where $\ell \geq 0$ and $S \cong \mathbb{Z}_4$ or $S \cong \left\{ \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2 \right\}$, then the unit graph $G(R)$ is a planar graph. This completes the proof of the theorem. □

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