A NOTE ON PERIODIC SOLUTIONS OF RICCATI EQUATIONS

M. R. MOKHTARZADEH, M. R. POURNAKI, AND A. RAZANI

Abstract. In this note, we show that under certain assumptions the scalar Riccati differential equation $x' = a(t)x + b(t)x^2 + c(t)$ with periodic coefficients admits at least one periodic solution. Also, we give two illustrative examples in order to indicate the validity of the assumptions.

1. Introduction

A large class of dynamical systems appearing throughout the field of engineering and applied mathematics is described by the second order differential equation of the form

$$y'' + p(t)y' + q(t)y = r(t), \tag{1}$$

where p, q, and r are real functions on $\mathbb{R}$. In general, there is no method for solving nonhomogeneous linear second order differential equations and, therefore, a complete analysis of (1) does not exist. Nevertheless, in the homogeneous case, when $r = 0$ in (1), by making the change of variable $x = -y'/y$, we are led to a first order differential equation of the form

$$x' = -p(t)x + x^2 + q(t). \tag{2}$$

Although the analysis of these kinds of differential equations are still in a preliminary stage, recently various issues concerning theoretical aspects of such differential equations have been successfully clarified.

In the literature, (2) is a special case of a more general one, so-called *scalar Riccati differential equation*, namely

$$x' = a(t)x + b(t)x^2 + c(t), \tag{3}$$

where a, b, and c are real functions on $\mathbb{R}$. The study of (3) has long been an important topic and dates from the early period of modern mathematical analysis. It began with examinations of particular cases of (1) by James Bernoulli (1654–1705) and then by Count Jacopo Francesco Riccati (1676–1754).

The generalization of scalar Riccati differential equation to the matrix case gives us *matrix Riccati differential equation* which is one of the central objects of present day

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control theory. In fact, in the theory of control systems, the qualitative control problem has received considerable research interests. This problem is regarded as an extension of the classical result of Kalman et al. [10] on controllability and stability of linear systems which is relevant to such differential equations (see [5, 16, 4, 3, 7, 8]). Matrix Riccati differential equations also play predominant roles in other control theory problems such as dynamic games, linear systems with Markovian jumps, and stochastic control. The study of such differential equations, which also appears in a number of other areas such as biomathematics and multidimensional transport processes, is an interesting area of current research. There exists a rather extensive literature on the matrix Riccati differential equation, mainly developed within the automatic control literature. We refer the readers to [3] as an extensive survey as well as to [5, 16, 4, 9] as fundamental papers on this area.

The analysis of periodic systems has long been a topic of interest. In this direction, an important question, which has been studied extensively by a number of authors (see, for example [1, 12, 14, 18, 17]), is whether Riccati differential equations can support periodic solutions or not. For example, in theoretical aspects, knowledge of the periodic solutions is important for understanding the phase portrait of the Riccati differential equations and, in particular, the qualitative behavior of solutions (see, for example [16]). On the other hand, on the applied side, in the problem of quadratic periodic optimization, arising for instance in the design of solar heating systems where the ambient temperature represents a periodic input, there occurs the need to compute the periodic solutions, if any, of a scalar or matrix Riccati differential equation with periodic coefficients. Another application is found in Kalman filtering of periodic systems such as orbiting satellites, seasonal phenomena like river flows, and econometric models, etc. We refer the readers to [2] for an overview on the structural properties of periodic systems, to [3] for the properties of periodic solutions to periodic Riccati differential equations, and to [6] for the study of the periodic Lyapunov differential equations. Also, the book by Reid [15] covers many areas in Riccati differential equations and is concerned with applications of these differential equations such as transmission line phenomena, theory of random processes, variational theory and optimal control theory, diffusion problems, and invariant imbedding.

In this note, we deal with scalar Riccati differential equations. In the light of the above discussion, it seems reasonable to consider (3) and asks when this differential equation has a periodic solution. In this direction, we assume that a, b, and c are ω-periodic continuous real functions on $\mathbb{R}$ and give certain conditions to guarantee the existence of at least one periodic solution for (3). The proof hinges on Schauder's fixed point theorem (see Theorem 2.2) applied to integral equation (5) which is a reformulation of (3). In order to indicate the validity of the assumptions made in our result, we also treat two illustrative examples.

2. Preliminaries

In this section, we present a brief survey of notions and results of functional analysis which we shall need later. The reader is referred to [11] for a fuller treatment of the subject.

We start by recalling the definition of a normed space. Let X be a vector space over $\mathbb{R}$. Then a *norm* on X is a function $\| \ \| : X \longrightarrow \mathbb{R}$ such that (1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$, (2) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$ and for all $x \in X$, (3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality). A vector space X over $\mathbb{R}$ together with a norm $\| \ \|$ is called a *normed space*. In this case, the *distance* from x to y in X is defined by $d(x, y) = \|x - y\|$. This defines a metric on X. Therefore, every normed space has a metric and so has an associated topology. All the standard topological notions same as open sets, closed sets, bounded sets, convergence, etc. may be applied to X. Also, we recall that in a normed space X, a subset A of X is called *convex* if $\alpha x + (1-\alpha)y \in A$ for all $x, y \in A$ and for all α with $0 \leq \alpha \leq 1$.

We are now going to define Banach spaces which are the most manageable among all types of normed spaces due to their metric structure. A normed space X is called a *Banach space* if it is complete, i.e., if every Cauchy sequence in X is convergent. Also, for a given Banach space X, a *compact* operator on X is a bounded operator $S : X \longrightarrow X$ that maps the unit ball in X to a set in X with compact closure. It is easy to see that the operator S is compact on X if and only if every bounded sequence $\{\phi_n\}$ on X has a subsequence $\{\phi_{n_i}\}$ such that $\{S(\phi_{n_i})\}$ is convergent on X.

We also recall that a given sequence $\{\phi_n(t)\}$ of functions from $[a, b]$ to $\mathbb{R}$, is called *equicontinuous* if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$ and for all $t_1, t_2 \in [a, b]$, $|t_1 - t_2| < \delta$ implies that $|\phi_n(t_1) - \phi_n(t_2)| < \epsilon$.

In the sequel, we also need the following weak version of Ascoli–Arzelà theorem.

Theorem 2.1 (Ascoli–Arzelà). *Let $\{\phi_n(t)\}$ be a sequence of functions from $[a, b]$ to $\mathbb{R}$ which is uniformly bounded and equicontinuous. Then $\{\phi_n(t)\}$ has a uniformly convergent subsequence.*

Finally, we close this section with the following fixed point theorem which is originally due to Schauder and is a key tool for proving the main result of this note (see Main Theorem 3.2).

Theorem 2.2 (Schauder). *Let X be a Banach space and Ω be a closed, bounded, and convex subset of X. If $S : \Omega \longrightarrow \Omega$ is a compact operator, then S has at least one fixed point on Ω.*

3. Main result

In this section, we state and prove the main contribution of this note (see Main Theorem 3.2). In order to do this, suppose that a, b, and c are ω-periodic continuous real functions on $\mathbb{R}$ such that $\int_0^\omega a(\xi)d\xi \neq 0$. Define the function $G : [0, \omega] \times [0, \omega] \longrightarrow$

$\mathbb{R}$ as follows:

$$G(t,s)=\begin{cases}\frac{1}{1-\exp(\int_0^\omega a(\xi)d\xi)}\exp\big(\int_s^t a(\xi)d\xi\big) & : \quad 0\le s\le t\le\omega,\\[2ex] \frac{\exp(\int_0^\omega a(\xi)d\xi)}{1-\exp(\int_0^\omega a(\xi)d\xi)}\exp\big(\int_s^t a(\xi)d\xi\big) & : \quad 0\le t\le s\le\omega.\end{cases} \tag{4}$$

The following lemma is useful for proving the Main Theorem 3.2.

Lemma 3.1. *Let a, b, and c be ω-periodic continuous real functions on $\mathbb{R}$ such that $\int_0^\omega a(\xi)d\xi\neq 0$. Suppose that x is a continuous real function on $\mathbb{R}$. If x is a solution of the integral equation*

$$x(t)=\int_0^\omega G(t,s)\big(b(s)x^2(s)+c(s)\big)ds, \tag{5}$$

then x is a solution of (3).

Proof. By assumption, x is a solution of integral equation (5). Therefore, by using expression (4), we may write

$$\begin{aligned}
x(t) &= \int_0^\omega G(t,s)\big(b(s)x^2(s)+c(s)\big)ds\\
&= \int_0^t G(t,s)\big(b(s)x^2(s)+c(s)\big)ds+\int_t^\omega G(t,s)\big(b(s)x^2(s)+c(s)\big)ds\\
&= \tfrac{1}{1-\exp(\int_0^\omega a(\xi)d\xi)}\int_0^t \exp\big(\textstyle\int_s^t a(\xi)d\xi\big)\big(b(s)x^2(s)+c(s)\big)ds\\
&\quad - \tfrac{\exp(\int_0^\omega a(\xi)d\xi)}{1-\exp(\int_0^\omega a(\xi)d\xi)}\int_\omega^t \exp\big(\textstyle\int_s^t a(\xi)d\xi\big)\big(b(s)x^2(s)+c(s)\big)ds\\
&= \tfrac{\exp(\int_0^t a(\xi)d\xi)}{1-\exp(\int_0^\omega a(\xi)d\xi)}\int_0^t \tfrac{1}{\exp(\int_0^s a(\xi)d\xi)}\big(b(s)x^2(s)+c(s)\big)ds\\
&\quad - \tfrac{\exp(\int_0^\omega a(\xi)d\xi)\exp(\int_0^t a(\xi)d\xi)}{1-\exp(\int_0^\omega a(\xi)d\xi)}\int_\omega^t \tfrac{1}{\exp(\int_0^s a(\xi)d\xi)}\big(b(s)x^2(s)+c(s)\big)ds.
\end{aligned} \tag{6}$$

Thus, we obtain

$$
\begin{aligned}
x'(t) = \; & \tfrac{a(t)\exp(\int_0^t a(\xi)d\xi)}{1-\exp(\int_0^\omega a(\xi)d\xi)} \int_0^t \tfrac{1}{\exp(\int_0^s a(\xi)d\xi)} \big(b(s)x^2(s) + c(s)\big) ds \\
& + \tfrac{1}{1-\exp(\int_0^\omega a(\xi)d\xi)} \big(b(t)x^2(t) + c(t)\big) \\
& - \tfrac{a(t)\exp(\int_0^\omega a(\xi)d\xi)\exp(\int_0^t a(\xi)d\xi)}{1-\exp(\int_0^\omega a(\xi)d\xi)} \int_\omega^t \tfrac{1}{\exp(\int_0^s a(\xi)d\xi)} \big(b(s)x^2(s) + c(s)\big) ds \\
& - \tfrac{\exp(\int_0^\omega a(\xi)d\xi)}{1-\exp(\int_0^\omega a(\xi)d\xi)} \big(b(t)x^2(t) + c(t)\big) \\
= \; & a(t)x(t) + b(t)x^2(t) + c(t),
\end{aligned}
\tag{7}
$$

which shows that x is a solution of (3) as requested. □

Note that expression (4) is, in fact, the Green's function of (3). Therefore, by using methods for finding Green's function, we may find the kernel $G(t, s)$ appeared in expression (4). However, our approach is different, but for going through the details of finding Green's function we refer the reader to [13].

We now state and prove the Main Theorem 3.2 which is the main contribution of this note.

Main Theorem 3.2. *Let a, b, and c be ω-periodic continuous real functions on $\mathbb{R}$ such that $\int_0^\omega a(\xi)d\xi \neq 0$. Consider*

$$M = \sup_{0\le t,s\le\omega} |G(t, s)|, \tag{8}$$

$$N = \sup_{0\le t\le\omega} \left| \int_0^\omega G(t, s)c(s)ds \right|, \tag{9}$$

and suppose that

$$\int_0^\omega |b(\xi)|d\xi \le \frac{1}{4MN}. \tag{10}$$

Then (3) *admits at least one ω-periodic solution.*

Proof. Let

$$X = \{\phi \mid \phi \text{ is a } \omega\text{-periodic continuous real function on } \mathbb{R}\} \tag{11}$$

and for $\phi \in X$ define $\|\phi\| = \sup_{0\le t\le\omega} |\phi(t)|$. It is easy to see that X is a Banach space. Define the function $\psi : [0, \omega] \longrightarrow \mathbb{R}$ as

$$\psi(t) = \int_0^\omega G(t, s)c(s)ds \tag{12}$$

and consider

$$\Omega = \{\phi \in X \mid \|\phi - \psi\| \leq N\}. \tag{13}$$

It is easy to see that Ω is closed, bounded, and convex subset of X. Define the operator $S : \Omega \longrightarrow X$ by sending ϕ to $S(\phi)$, where $S(\phi)$ defined as

$$S(\phi)(t) = \int_0^\omega G(t,s)\big(b(s)\phi^2(s) + c(s)\big)ds. \tag{14}$$

First, we claim that S maps Ω into Ω. In order to show this, by using (13) and (12), we obtain $|\phi(t)| \leq N + |\psi(t)| \leq 2N$ holds for all $\phi \in \Omega$ and for all $t \in [0,\omega]$. Therefore, (14), (12), (8), and (10) imply that for all $\phi \in \Omega$ and for all $t \in [0,\omega]$,

$$\begin{aligned} |S(\phi)(t) - \psi(t)| = &\ |\textstyle\int_0^\omega G(t,s)b(s)\phi^2(s)ds| \\ \leq &\ 4MN^2 \textstyle\int_0^\omega |b(s)|ds \\ \leq &\ N. \end{aligned} \tag{15}$$

Thus, for all $\phi \in \Omega$, we have $\|S(\phi) - \psi\| \leq N$ and so $S(\phi) \in \Omega$. This shows that S is an operator from Ω into Ω.

Next, we show that S is compact. In order to do this, suppose $\{\phi_n\}$ is a sequence on Ω which is, by (13), bounded. Thus, there exists $L > 0$ such that for all $n \in \mathbb{N}$ and for all $t \in [0,\omega]$, we have $|\phi_n(t)| \leq L$. We should show that $\{\phi_n\}$ has a subsequence, say $\{\phi_{n_i}\}$, such that $\{S(\phi_{n_i})\}$ is convergent on Ω. Note that, by Lemma 3.1, for all $n \in \mathbb{N}$, the function $S(\phi_n)$ is, in fact, differentiable and for all $t \in [0,\omega]$ we have

$$S(\phi_n)'(t) = a(t)\phi_n(t) + b(t){\phi_n}^2(t) + c(t). \tag{16}$$

Therefore, for all $n \in \mathbb{N}$ and for all $t \in [0,\omega]$, we have $|S(\phi_n)'(t)| \leq AL + BL^2 + C$, where A, B, and C are the maximum values of $|a|$, $|b|$, and $|c|$ on $[0,\omega]$, respectively. Therefore, for given $\varepsilon > 0$, if we consider $\delta = \varepsilon/(AL + BL^2 + C)$, then for all $n \in \mathbb{N}$ and for all $t_1, t_2 \in [0,\omega]$, $|t_1 - t_2| < \delta$ implies that

$$|S(\phi_n)(t_1) - S(\phi_n)(t_2)| \leq (AL + BL^2 + C)|t_1 - t_2| < \varepsilon. \tag{17}$$

Thus, $\{S(\phi_n(t))\}$ as a sequence of functions on $[0,\omega]$ is equicontinuous and Theorem 2.1 then implies that there exists a subsequence of $\{S(\phi_n(t))\}$, say $\{S(\phi_{n_i}(t))\}$, which is uniformly convergent on $[0,\omega]$. This means that $\{S(\phi_{n_i})\}$ is convergent on Ω and so S is compact.

Therefore, Theorem 2.2 implies that there exists $x \in \Omega$ such that $S(x) = x$, i.e., for all $t \in [0,\omega]$,

$$x(t) = \int_0^\omega G(t,s)\big(b(s)x^2(s) + c(s)\big)ds. \tag{18}$$

Since $x \in \Omega$, x is a ω-periodic continuous real function on $\mathbb{R}$ and so what remains to be proved is that x is indeed a solution to (3). But this already proven by Lemma 3.1. $\square$

4. Two illustrative examples

In this section, we treat the following two illustrative examples. The first example is generated by trial and error process, using computer codes in Mathematica 5.2 with symbolic operations. Therefore, it seems to be quite far from a real practical problem. However, it shows that the novelty of our approach, since the previous results in the literature are inapplicable for proving the existence of periodic solutions of it. Also, this example indicates the validity of the assumptions made in the Main Theorem 3.2. These are the only justification for putting this example. Instead, the second example comes from mathematical modeling of natural phenomenons.

Example 4.1. Suppose that a, b, and c are 1-periodic continuous real functions on $\mathbb{R}$ which are defined as follows:

$$a(t) = \frac{\pi(447\cos\pi t - 288\cos 2\pi t + \cos 3\pi t + 23762\sin\pi t + 288\sin 2\pi t - 4\sin 3\pi t + 480)}{2(\cos\pi t + 3)(136\cos\pi t + 4\cos 2\pi t - 312\sin\pi t + \sin 2\pi t - 8063)}, \tag{19}$$

$$b(t) = \frac{-48\pi(3\cos\pi t + 53\sin\pi t + 1)}{-15306\cos\pi t + 160\cos 2\pi t + 4\cos 3\pi t - 1871\sin\pi t - 306\sin 2\pi t + \sin 3\pi t - 48242}, \tag{20}$$

$$c(t) = \frac{\pi(303\cos\pi t - 288\cos 2\pi t + \cos 3\pi t + 21218\sin\pi t + 288\sin 2\pi t - 4\sin 3\pi t + 432)}{15306\cos\pi t - 160\cos 2\pi t - 4\cos 3\pi t + 1871\sin\pi t + 306\sin 2\pi t - \sin 3\pi t + 48242}. \tag{21}$$

Here, we have $\int_0^1 a(\xi)d\xi = -1.013501$, and so the function G is as follows:

$$G(t,s) = \begin{cases} 1.56972\exp\left(\int_s^t a(\xi)d\xi\right) & : \quad 0 \le s \le t \le 1, \\ 0.569725\exp\left(\int_s^t a(\xi)d\xi\right) & : \quad 0 \le t \le s \le 1. \end{cases} \tag{22}$$

Since

$$M = \sup_{0\le t,s\le 1} |G(t,s)| = 1.56972 \tag{23}$$

and

$$N = \sup_{0\le t\le 1}\left|\int_0^1 G(t,s)c(s)ds\right| = 1.37309, \tag{24}$$

we have

$$\int_0^1 |b(\xi)|d\xi = 0.107581 < 0.115988 = \frac{1}{4MN}. \tag{25}$$

Therefore, the Main Theorem 3.2 implies that (3) admits at least one 1-periodic solution. This 1-periodic solution may be given by expression (29), where k is an arbitrary constant. Note that expression (29) is the general solution of (3) defined by (19), (20), and (21). The general solution is generated by knowing three particular solutions (26), (27), and (28) of the equation. Here, all numerical results are correct to 50 digits, using arbitrary precision facilities devised in this software, and then all results truncated to 6 decimal places. For sacking more accurate results, we also normalized variable and restricted ourselves to the interval $[0,1)$.

$$x_1(t) = 1, \tag{26}$$

$$x_2(t) = 2 + \tfrac{1}{3}\cos\pi t, \tag{27}$$

$$x_3(t) = 8 + \tfrac{1}{8}(\cos\pi t + \sin\pi t), \tag{28}$$

$$x(t) = \tfrac{k\cos 2\pi t + 12k\sin\pi t + k\sin 2\pi t + 673k + 2(62k+5)\cos\pi t - 6\sin\pi t - 288}{2(3k+5)\cos\pi t + 6(8(7k-6) + (k-1)\sin\pi t)}. \tag{29}$$

Example 4.2. Consider the Riccati differential equation

$$x' = x + b(t)x^2 + \sin t, \tag{30}$$

where b is a 2π-periodic continuous real function on $\mathbb{R}$. This equation comes from mathematical modeling of natural phenomenons. Here, we have $\int_0^{2\pi} d\xi = 2\pi$, and so the function G is as follows:

$$G(t,s) = \begin{cases} \frac{1}{1-\exp(2\pi)}\exp(t-s) & : \quad 0 \le s \le t \le 2\pi, \\ \\ \frac{\exp(2\pi)}{1-\exp(2\pi)}\exp(t-s) & : \quad 0 \le t \le s \le 2\pi. \end{cases} \tag{31}$$

We have

$$M = \sup_{0 \le t,s \le 2\pi} |G(t,s)| = 1.001870, \tag{32}$$

$$N = \sup_{0 \le t \le 2\pi} \left| \int_0^{2\pi} G(t,s)(\sin s)ds \right| = 1.103237, \tag{33}$$

and so if

$$\int_0^{2\pi} |b(\xi)| d\xi \le \frac{1}{4MN} = 0.226182, \tag{34}$$

then the Main Theorem 3.2 implies that (30) admits at least one 2π-periodic solution. For instance, if we consider

$$b(t) = -\frac{28(784 + \cos t + 29\sin t)}{(784 + \cos t)^2}, \tag{35}$$

then since

$$\int_0^{2\pi} |b(\xi)| d\xi = 0.2244 < 0.226182, \tag{36}$$

we obtain that (30) admits at least one 2π-periodic solution, that is,

$$x(t) = \frac{1}{28}(784 + \cos t). \tag{37}$$

5. Conclusion

In this note, we investigate the existence of periodic solutions for a class of scalar Riccati differential equations. Suppose that a, b, and c are ω-periodic continuous real functions on $\mathbb{R}$ such that $\int_0^\omega a(\xi)d\xi \neq 0$. Consider

$$M = \sup_{0 \leq t, s \leq \omega} |G(t,s)|, \tag{38}$$

$$N = \sup_{0 \leq t \leq \omega} \left| \int_0^\omega G(t,s)c(s)ds \right|, \tag{39}$$

and suppose that

$$\int_0^\omega |b(\xi)|d\xi \leq \frac{1}{4MN}, \tag{40}$$

where the kernel $G(t,s)$ is defined as follows:

$$G(t,s) = \begin{cases} \frac{1}{1-\exp(\int_0^\omega a(\xi)d\xi)} \exp\left(\int_s^t a(\xi)d\xi\right) & : \quad 0 \leq s \leq t \leq \omega, \\ \frac{\exp(\int_0^\omega a(\xi)d\xi)}{1-\exp(\int_0^\omega a(\xi)d\xi)} \exp\left(\int_s^t a(\xi)d\xi\right) & : \quad 0 \leq t \leq s \leq \omega. \end{cases} \tag{41}$$

Then the scalar Riccati differential equation $x' = a(t)x + b(t)x^2 + c(t)$ admits at least one ω-periodic solution. In order to indicate the validity of the assumptions made in our result, we also treat two illustrative examples.

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M. R. Mokhtarzadeh, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
E-mail address: mrmokhtarzadeh@ipm.ir

M. R. Pournaki, Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
E-mail address: pournaki@ipm.ir
URL: http://math.ipm.ac.ir/pournaki/

A. Razani, Department of Mathematics, Faculty of Science, Imam Khomeini International University, P.O. Box 34194-288, Qazvin, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
E-mail address: razani@ikiu.ac.ir
URL: http://math.ipm.ac.ir/razani/