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Graphs Attached to Rings Revisited

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Abstract In this paper, we discuss some recent results on graphs attached to rings. In particular, we deal with comaximal graphs, unit graphs, and total graphs. We then define the notion of cototal graph and, using this graph, we characterize the rings which are additively generated by their zero divisors. Finally, we glance at graphs attached to other algebraic structures.

Keywords Chromatic index · Chromatic number · Clique number · Comaximal graph · Connectedness · Cototal graph · Counit graph · Diameter · Finite ring · Girth · Hamiltonian cycle · Hamiltonian graph · Planarity · Total graph · Unit element · Unit graph · Weakly perfect graph · Zero-divisor graph

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الملخص

في هذه الورقة البحثية، نناقش بعض النتائج الحديثة عن البيانات المرفقة بالحلقات. بشكل خاص، نتعامل مع البيانات الأعظمية المرفقة، وبيانات الوحدة، والبيانات الإجمالية. ثم نعرِّف مفهوم البيان الإجمالي المرفق، وباستعمال هذا البيان، نميز الحلقات المولَّدة جمعيًا بوساطة قواسم الصفر. في النهاية، نلقي نظرة سريعة على البيانات المرفقة ببناءات جبرية أخرى.

1 Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vise versa. Associating a graph with an algebraic structure is a research subject in this area and has attracted considerable attention. In fact, research in this subject aims at exposing the relationship between algebra and graph theory and at advancing applications of one to the other. The story goes back to a paper of Beck [8] in 1988, where he introduced the idea of a zero-divisor graph of a commutative ring R with identity. He defined $\Gamma_0(R)$ to be the graph whose vertices are elements of R and in which two vertices x and y are adjacent if and only if $xy = 0$. He was mostly concerned with coloring $\Gamma_0(R)$. Let $\chi(R)$ and $\omega(R)$ denote the chromatic number and the clique number of $\Gamma_0(R)$, respectively. Beck conjectured that $\chi(R) = \omega(R)$. Such graphs are called weakly perfect graphs. This investigation of coloring of a commutative ring was then continued by Anderson and Naseer [2]. They gave a counterexample to the above conjecture of Beck. A different method of associating a zero-divisor graph to a commutative ring R was proposed by Anderson and Livingston [5]. They believed that this better illustrated the zero-divisor structure of the ring. They defined $\Gamma(R)$ to be the graph whose vertices are nonzero zero divisors of R and in which two vertices x and y are adjacent if and only if $xy = 0$. This graph is defined slightly differently than the graph introduced by Beck who took the set of vertices to be all of R. For a survey and recent results concerning zero-divisor graphs, we refer the reader to [3].

One can associate other graphs to rings or other algebraic structures. The benefit of studying these graphs is that one may find some results about the algebraic structures and vise versa. There are three major problems in this area: (1) characterization of the resulting graphs, (2) characterization of the algebraic structures with isomorphic graphs, and (3) realization of the connections between the structures and the corresponding graphs. This is a particularly interesting subject for some graph theorists and algebraists, since it relates two very different areas of mathematics, involves fun computations, and can be studied at many different levels of mathematical expertise and sophistication.

In this survey, we discuss some recent results on graphs attached to rings. In particular, we deal with comaximal graphs, unit graphs, and total graphs. We then define the notion of cototal graph and using this graph we characterize the rings which are additively generated by their zero divisors. Finally, we glance at graphs attached to other algebraic structures.

In what follows, all rings are associative with nonzero identity, which is preserved by ring homomorphisms and inherited by subrings. Also, throughout the paper, by a graph we mean a finite undirected graph without loops or multiple edges.

2 Comaximal Graphs

Let G be a graph. A *k-coloring* of the vertices of G is an assignment of k colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number k such that G admits a k-coloring. A *clique* in G is a set of pairwise adjacent vertices of G. A clique of maximum size is called a *maximum clique*. The *clique number* of G, denoted by $\omega(G)$, is the number of vertices of a maximum clique in G. The parameters $\chi(G)$ and $\omega(G)$ have been extensively studied by many authors. Nevertheless, finding the chromatic number and the clique number of a graph is long known to be an NP-hard problem (cf. [12]). A graph G is called *weakly perfect* provided $\chi(G) = \omega(G)$. It is known that deciding whether $\chi(G) = \omega(G)$ is an NP-complete problem (cf. [26]).

2.1 Comaximal Graphs: Definition and Remarks

Let R be a commutative ring and let $J(R)$ denote the Jacobson radical of R. In 1995, Sharma and Bhatwadekar [32] introduced the *comaximal graph* of R, denoted by $\Gamma(R)$, which is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if

$Rx + Ry = R$. It is easy to see that the unit elements of R are adjacent to all elements of R, while all elements of $J(R)$ are adjacent only to the unit elements. In [32], the authors have proved that $\chi(R) < \infty$ if and only if R is a finite ring, and in this case, they have shown that the graph is weakly perfect. In fact, they showed that $\chi(R) = \omega(R) = t + \ell$, where t and ℓ denote the number of maximal ideals of R and the number of unit elements of R, respectively. Later, Maimani et al. [24] studied $\Gamma(R)$ and its two subgraphs $\Gamma_1(R)$ and $\Gamma_2(R)$, which are induced by the unit elements and the nonunit elements of R, respectively.

2.2 Basic Properties of Comaximal Graphs

For r a nonnegative integer, an *r-partite graph* is one whose vertex set is partitioned into r disjoint parts in such a way that the two end vertices for each edge lie in distinct partitions. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same partition. The (complete) two-partite graph is called the (*complete*) *bipartite graph*. The complete bipartite graph with exactly two partitions of size m and n is denoted by $K_{m,n}$, and $K_{1,1}$ is called the *star graph*.

For a graph G, let $V(G)$ denote the set of vertices, and let $E(G)$ denote the set of edges. Let G be graph, and suppose $x, y \in V(G)$. We recall that a *walk* between x and y is a sequence $x = v_0, e_1, v_1, \ldots, e_k, v_k = y$ of vertices and edges of G, denoted by

$$x = v_0 \xrightarrow{e_1} v_1 \longrightarrow \cdots \xrightarrow{e_k} v_k = y,$$

such that for every i with $1 \le i \le k$, the edge e_i has endpoints v_{i-1} and v_i. Also, a *path* between x and y is a walk between x and y without repeated vertices. A *trail* between x and y is a walk between x and y in which all the edges are distinct. A graph G is called *connected* if for any vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected*. A graph G is said to be *totally disconnected* if $E(G) = \emptyset$. For a graph G and vertices x and y of G, the *distance* between x and y, denoted by $d(x, y)$, is the number of edges in a shortest path between x and y. If there is no path between x and y, then we write $d(x, y) = \infty$. The largest distance among all distances between pairs of the vertices of a graph G is called the *diameter* of G and is denoted by diam (G).

In the following, we denote the set of maximal ideals of the ring R by $\mathrm{Max}(R)$. Note that in Theorem 2.2(b), Theorem 2.3, and the last part of Theorem 2.4, the assertion $|\mathrm{Max}(R)| \ge 2$ has an important role and without it the results are wrong. Unfortunately, this assertion missed the point in the statement of Theorem 2.2(b) in the published version (cf. [24, Proposition 2.3]).

Theorem 2.1 ([24], Theorem 2.2) *Let R be a commutative ring. Then the following statements are equivalent:*

(a) *The graph $\Gamma_2(R)\backslash J(R)$ is completely bipartite.*
(b) $|\mathrm{Max}(R)| = 2$.

Theorem 2.2 ([24], Proposition 2.3) *Let R be a commutative ring and let $n > 1$. Then the following statements hold:*

(a) *If $|\mathrm{Max}(R)| = n < \infty$, then the graph $\Gamma_2(R)\backslash J(R)$ is n-partite.*
(b) *If $|\mathrm{Max}(R)| \ge 2$ and the graph $\Gamma_2(R)\backslash J(R)$ is n-partite, then $|\mathrm{Max}(R)| \le n$.*
In this case, if the graph $\Gamma_2(R)\backslash J(R)$ is not $(n-1)$-partite, then $|\mathrm{Max}(R)| = n$.

Theorem 2.3 ([24], Proposition 2.4) *Let R be a commutative ring with $|\mathrm{Max}(R)| \ge 2$. Then the following statements hold:*

(a) *If the graph $\Gamma_2(R)\backslash J(R)$ is complete n-partite, then $n = 2$.*
(b) *If there exists a vertex of the graph $\Gamma_2(R)\backslash J(R)$, which is adjacent to its every other vertex, then $R \cong \mathbb{Z}_2 \times F$, where F is a field.*

Theorem 2.4 ([24], Theorem 3.1, Lemma 3.2, and Proposition 3.3) *Let R be a commutative ring. Then the graph $\Gamma_2(R)\backslash J(R)$ is connected and we have* $\mathrm{diam}(\Gamma_2(R)\backslash J(R)) \le 3$. *Moreover,* $\mathrm{diam}(\Gamma_2(R)\backslash J(R)) = 1$ *if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also if $|\mathrm{Max}(R)| \ge 2$, then* $\mathrm{diam}(\Gamma_2(R)\backslash J(R)) = 2$ *if and only if one of the following statements hold:*

(a) *The Jacobson radical $J(R)$ is a prime ideal of R.*
(b) $|\mathrm{Max}(R)| = 2$ *and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

A ring R is called *clean* if every element of R is the sum of an idempotent and a unit, and R is called *uniquely clean* if this representation is unique (cf. [1,28]). This definition dates back to 1977 when it was introduced by Nicholson [28]. For example, quasi-local rings are clean. The following result gives an application of Sharma–Bhatwadekar graph to characterize clean rings.

Theorem 2.5 ([24], Theorem 2.5) *Let R be a commutative ring. Then the following statements are equivalent:*

(a) *The ring R is a finite product of quasi-local rings.*
(b) *The ring R is clean and $\omega(\Gamma_2(R)\backslash J(R))$ is finite.*

Theorem 2.6 ([24], Theorem 4.4) *Let $\{(R_i, \mathfrak{m}_i)\}_{i=1}^{m}$ and $\{(S_j, \mathfrak{n}_j)\}_{j=1}^{n}$ be two families of finite commutative quasi-local rings and let $R = R_1 \times \cdots \times R_m$ and $S = S_1 \times \cdots \times S_n$. If $\Gamma(R) \cong \Gamma(S)$, then we have $m = n$ and there is a permutation σ on the set $\{1, 2, \ldots, m\}$ such that $|R_i/\mathfrak{m}_i| = |S_{\sigma(i)}/\mathfrak{n}_{\sigma(i)}|$ and $R_i/\mathfrak{m}_i \cong S_{\sigma(i)}/\mathfrak{n}_{\sigma(i)}$, for every $i = 1, \ldots, m$. In particular, if $\Gamma(R) \cong \Gamma(S)$ and every R_i is a finite field, then $R \cong S$.*

In addition, Maimani et al. in Proposition 4.7. of [24] have claimed that for given rings R and S, $\Gamma(R) \cong \Gamma(S)$ implies that $R/J(R) \cong S/J(S)$, where $J(R)$ and $J(S)$ denote the Jacobson radical of the rings R and S, respectively. In the case that the rings R and S are finite, this result indeed follows from Theorem 2.6 (cf. [24, Theorem 4.4]), but in general this is not true. The authors would like to thank S.M. Moconja and Z.Z. Petrović for pointing this out and giving a counterexample (cf. [27, Example 4.1]).

A *cycle* of a graph is a path such that the start and end vertices are the same. The number of edges in a cycle is called its *length*. For a graph G, the *girth* of G is the length of a shortest cycle in G and is denoted by $\mathrm{gr}(G)$. If G has no cycles, we define the girth of G to be infinite, and in this case, G is called a *forest*. A connected graph is *Eulerian* if there exists a closed trail containing every edge.

Theorem 2.7 ([33], Theorem 3.5) *Let R be a commutative ring. Then the following are equivalent for $\Gamma_2(R)\backslash J(R)$:*

(i) *$\Gamma_2(R)\backslash J(R)$ is a forest.*
(ii) *$\Gamma_2(R)\backslash J(R)$ is either totally disconnected or a star graph.*
(iii) *R is either a local ring which is not a field or R is isomorphic to $\mathbb{Z}_2 \times F$, where F is a field.*

Theorem 2.8 ([33], Theorem 3.10) *Let $R \cong R_1 \times R_2 \times \cdots \times R_k$ be a finite commutative ring, where R_i is a finite local ring for every i and $k \geq 2$. Then the following are equivalent:*

(i) *$\Gamma_2(R)\backslash J(R)$ is Eulerian.*
(ii) *$R \ncong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and $R \ncong R_1 \times \cdots \times R_t \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $|R_i|$ is odd for $i \leq t$ and $1 \leq t < k$.*
(iii) *$|R_i| = 2r$ with $r \geq 2$ for some i, or $|R_i|$ is odd for every i.*

A graph G is said to be *planar* if it can be drawn in the plane in such a way that its edges intersect only at their ends. In the following theorem, $\mathbb{F}_4$ denotes the field with four elements.

Theorem 2.9 ([33], Corollary 5.3) *Let R be a finite commutative ring. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2), \mathbb{F}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

It is also shown that the clique number of $\Gamma_2(R)$ is equal to the number of maximal ideals of R (cf. [33]). In addition, Wang [34] considered the comaximal graph $\Gamma(R)$ of a not necessarily commutative ring R. He showed that if R is Artinian, then $\Gamma(R)\backslash J(R)$ is connected. Also, for R the full ring of all $n \times n$ matrices over a finite field F, the minimal degree, maximal degree, clique number, connectivity and chromatic number of $\Gamma(R)$ are computed. For a short survey concerning zero-divisor graphs, as well as comaximal graphs, we refer the reader to [31].

3 The Story Behind Unit Graphs

Let p be a prime number and let k be a positive integer. A while ago, we (H. R. Maimani, M. R. Pournaki, and S. Yassemi) defined a graph, denoted by $G(p^k)$, based on positive integers and their sums. The vertices of $G(p^k)$ are the elements of $\{1, \ldots, p^k\}$ and distinct vertices x and y are defined to be adjacent if and only if $\gcd(x + y, p) = 1$. We then proved that the class of these graphs is a new class of weakly perfect graphs.

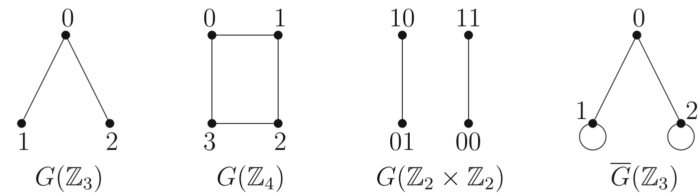

Fig. 1 The unit graphs of some specific rings

More precisely, we showed that for a given prime number p and for a positive integer k, the graph $G(p^k)$ is a weakly perfect graph (cf. [20, Theorem 2.1]). Moreover, we proved that

$$\chi(G(p^k)) = \omega(G(p^k)) = \begin{cases} 2 & \text{if } p = 2, \\ p^{k-1}(p-1)/2 + 1 & \text{if } p > 2. \end{cases}$$

After a while, we realized that these graphs have been already included in a paper of Grimaldi [14]. He has denoted these graphs by $G(\mathbb{Z}_{p^k})$ instead of $G(p^k)$ and has found some of their properties. Let n be a positive integer, and let $\mathbb{Z}_n$ be the ring of integers modulo n. In 1989, Grimaldi defined a graph $G(\mathbb{Z}_n)$ based on the elements and units of $\mathbb{Z}_n$. The vertices of $G(\mathbb{Z}_n)$ are the elements of $\mathbb{Z}_n$; distinct vertices x and y are defined to be adjacent if and only if $x + y$ is a unit of $\mathbb{Z}_n$. For a positive integer m, it follows that $G(\mathbb{Z}_{2m})$ is a $\varphi(2m)$-regular graph, where φ is the Euler phi function. In case $m \geq 2$, $G(\mathbb{Z}_{2m})$ can be expressed as the union of $\varphi(2m)/2$ *Hamiltonian cycles*, i.e., cycles containing all the vertices of the graph. The odd case is not quite so easy, but the structure is clear and the results are similar to the even case. We recall that a *cone* over a graph is obtained by taking the categorical product of the graph and a path with a loop at one end, and then identifying all the vertices whose second coordinate is the other end of the path. When p is an odd prime, $G(\mathbb{Z}_p)$ can be expressed as a cone over a complete partite graph with $(p-1)/2$ partitions of size two. This leads to an explicit formula for the *chromatic polynomial* $p(t)$ of $G(\mathbb{Z}_p)$. Here, $p(t)$ is the number of ways needed to color the vertices of $G(\mathbb{Z}_p)$ using at most t colors in such a way that no two adjacent vertices receive the same color. The chromatic polynomial of a graph was introduced by Birkhoff in 1912 to study the four-color problem (cf. [9]). The article [14] also concludes with some properties of the graphs $G(\mathbb{Z}_{p^k})$, where p is a prime number and $k \geq 2$.

Based on that mentioned above, we decided to generalize $G(\mathbb{Z}_n)$ to $G(R)$, the unit graph of R, where R is an arbitrary associative ring with nonzero identity. This graph is a subgraph of the comaximal graph studied in Sect. 2 (also cf. [24]). But a question may occur: what is the benefit of the unit graphs? We showed that the unit graphs associated with rings can play a role in the study of rings. In fact, by using unit graphs, we characterized the rings which are generated by their units (cf. [21, Theorem 1.1]). More precisely, let R be a finite commutative ring with nonzero identity. Then R is generated by its units if and only if R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient. As a byproduct, if R is generated by its units, then every element of R can be written as a sum of at most three units.

3.1 Unit Graphs: Definition and Remarks

Let R be a ring and $U(R)$ be the set of unit elements of R. The *unit graph* of R, denoted by $G(R)$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x + y \in U(R)$. If we omit the word distinct, we obtain the *closed unit graph* denoted by $\overline{G}(R)$; this graph may have loops. Note that if $2 \notin U(R)$, then $\overline{G}(R) = G(R)$ (Fig. 1).

It is easy to see that, for given rings R and S, if $R \cong S$ as rings, then $G(R) \cong G(S)$ as graphs. This point is illustrated in Fig. 2 for the unit graphs of two isomorphic rings $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$.

3.2 Basic Properties of Unit Graphs

Let G_1 and G_2 be two vertex-disjoint graphs. The *category product* of G_1 and G_2 is denoted by $G_1 \dot{\times} G_2$. That is, $V(G_1 \dot{\times} G_2) := V(G_1) \times V(G_2)$; two distinct vertices (x, y) and (x', y') are adjacent if and only if x is adjacent to x' in G_1 and y is adjacent to y' in G_2. Clearly, we have $\overline{G}(R_1) \dot{\times} \overline{G}(R_2) \cong G(R_1 \times R_2)$. In Fig. 3, we illustrate the above point for the direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

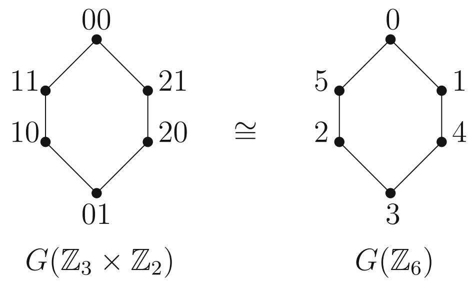

Fig. 2 The unit graphs of two isomorphic rings

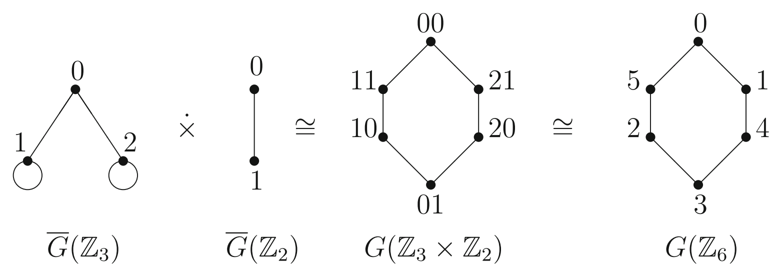

Fig. 3 The category product of two closed unit graphs

For a graph G and vertex $x \in V(G)$, the *degree* of x, denoted by $\deg(x)$, is the number of edges of G incident to x. For a nonnegative integer r, the graph G is called *r-regular* if the degree of each vertex of G is equal to r.

Theorem 3.1 ([6], Proposition 2.4) *Let R be a finite ring. Then the following statements hold for the unit graph of R:*

(a) *If $2 \notin U(R)$, then the unit graph $G(R)$ is a $|U(R)|$-regular graph.*
(b) *If $2 \in U(R)$, then for every $x \in U(R)$ we have* $\deg(x) = |U(R)| - 1$, *and for every $x \in R \backslash U(R)$ we have* $\deg(x) = |U(R)|$.

We point out that if $\{G_i\}_{i \in I}$ is a family of edge-disjoint subgraphs of a graph G such that $E(G) = \cup_{i \in I} E(G_i)$, then we may write $G = \oplus_{i \in I} G_i$. In this case, if $G_i \cong H$ for every $i \in I$, then we write $G = \oplus_I H$.

Theorem 3.2 ([6], Proposition 2.8) *Let R be a finite commutative ring and suppose that $J(R)$ denotes the Jacobson radical of R. Then the following statements hold:*

(a) *If $2 \notin U(R)$, then $G(R) \cong \oplus_{J(R)^2} G(R/J(R))$.*
(b) *If $2 \in U(R)$, then $G(R) \cong (\oplus_{J(R)^2} G(R/J(R))) \oplus (\oplus_{U(R)} K_{|J(R)|})$.*

The next result involves the Nagata extension of a ring, and so we recall this notion here. Let R be a commutative ring and let M be a unitary R-module. We may turn the cartesian product $R \times M$ into a ring by defining the operations $(r, x) + (r', x') := (r + r', x + x')$ and $(r, x)(r', x') := (rr', rx' + r'x)$. This ring is called the *Nagata extension* of R by M and is denoted by $R \oplus M$.

Theorem 3.3 ([6], Proposition 2.9) *Let R be a commutative ring and let M be a finite unitary R-module. Then the following statements hold:*

(a) *If $2 \notin U(R)$, then $G(R \oplus M) \cong \oplus_{M^2} G(R)$.*
(b) *If $2 \in U(R)$, then $G(R \oplus M) \cong (\oplus_{M^2} G(R)) \oplus (\oplus_{U(R)} K_{|M|})$.*

3.3 Characterizations of Certain Unit Graphs

A *complete graph* is a graph in which each pair of distinct vertices is joined by an edge. We denote the complete graph with n vertices by K_n. A *cycle graph* is a graph that consists of a single cycle.

Theorem 3.4 ([6], Theorem 3.2) *Let R be a finite ring. Then the unit graph $G(R)$ is a cycle graph if and only if R is isomorphic to one of the following rings:*

(a) $\mathbb{Z}_4$,
(b) $\mathbb{Z}_6$,
(c) $\left\{\left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2\right\}$.

By applying Corollary 4.5 of [11] we obtain that $G(R)$ is a 2-regular graph if and only if $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times S$, where S is one of the rings appearing in the statement of Theorem 3.4.

Theorem 3.5 ([6], Theorem 3.4) *Let R be a ring. Then the unit graph $G(R)$ is a complete graph if and only if R is a division ring with* $\mathrm{Char}(R) = 2$, *where* $\mathrm{Char}(R)$ *denotes the characteristic of R.*

The following theorem characterizes the complete bipartite unit graphs of rings.

Theorem 3.6 ([6], Theorem 3.5) *Let R be a commutative ring and $\mathfrak{m}$ be a maximal ideal of R such that $|R/\mathfrak{m}| = 2$. Then, $G(R)$ is a bipartite graph. Moreover, the unit graph $G(R)$ is a complete bipartite graph if and only if R is a local ring.*

In [29], the authors proved that the ring R is uniquely clean if and only if $|R/\mathfrak{m}| = 2$ for each maximal ideal $\mathfrak{m}$ of R. By combining this observation with Theorem 3.6, we conclude that if, for example, R is one of the following rings, then the unit graph $G(R)$ is bipartite.

$R = \mathbb{Z}_n$ for even n.
R is a uniquely clean ring.
$R = S \times S'$, where S is a uniquely clean ring and S' is an arbitrary ring.

Also note that if n is an integer such that $n = 2^m$, then $G(\mathbb{Z}_n)$ is a complete bipartite graph.

Let G_1 and G_2 be two graphs. The *join* of G_1 and G_2, which is denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1),\ y \in V(G_2)\}$. If we omit a perfect matching from a complete graph K_{2n}, then the resulting graph is called a *cocktail party* and denoted by $\mathrm{CP}(2n)$.

Theorem 3.7 ([6], Theorem 3.8) *Let R be a ring, n be a positive integer, and let p be an odd prime. Then, $G(R) \cong K_1 \vee \mathrm{CP}(p^n - 1)$ if and only if R is a field with p^n elements.*

Note that in the above theorem, if R is a field with $\mathrm{Char}(R) \neq 2$, then the unit graph $G(R)$ is isomorphic to $K_1 \vee H$, where H is an infinite cocktail party.

3.4 Connectedness of Unit Graphs

The unit graph $G(\mathbb{Z}_2 \times \mathbb{Z}_2)$, shown in Fig. 1, is not connected. There are also many other rings which have disconnected unit graphs. For example, suppose that $R = S \times T$ where $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and T is a ring. Then $G(R)$ is not connected. Thus unit graphs are not connected, in general.

Let R be a ring and k be a positive integer. An element $r \in R$ is said to be *k-good* if we may write $r = u_1 + \cdots + u_k$, where $u_1, \ldots, u_k \in U(R)$. The ring R is said to be *k-good* if every element of R is k-good. Following [13], we now define an invariant of a ring, called the *unit sum number*, which expresses, in a fairly precise way, how the units generate the ring. The *unit sum number* of R, denoted by $\mathbf{u}(R)$, is given by:

(a) $\min\{k \mid R \text{ is } k\text{−good}\}$ if R is k-good for some $k \geq 1$,
(b) ω if R is not k-good for every k, but every element of R is k-good for some k (that is, when at least $U(R)$ generates R additively),
(c) ∞ otherwise (that is, when $U(R)$ does not generate R additively).

For example, let D be a division ring. If $|D| \geq 3$, then $\mathbf{u}(D) = 2$; whereas if $|D| = 2$, that is, $D = \mathbb{Z}_2$ the field of two elements, then $\mathbf{u}(\mathbb{Z}_2) = \omega$. We have also $\mathbf{u}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \infty$ (cf. [7] for unit sum numbers of some other rings).

Theorem 3.8 ([6], Theorem 4.3) *Let R be a ring. Then the unit graph $G(R)$ is connected if and only if $\mathbf{u}(R) \leq \omega$.*

Note that there are infinitely many rings R such that $\mathbf{u}(R) \leq \omega$ and therefore, by Theorem 3.8, their unit graphs are connected.

We recall that for a graph G, a subset S of the vertex set of G is called a *dominating set* if every vertex not in S is adjacent to a vertex in S.

Theorem 3.9 ([6], Corollary 4.4) *Let R be a ring. If $U(R)$ is a dominating set of $G(R)$, then the unit graph $G(R)$ is connected.*

Note that there are infinitely many rings R such that $U(R)$ is a dominating set of $G(R)$ and, therefore, by Theorem 3.9, the unit graph $G(R)$ is connected. For example, if R is a 2-good ring, then $U(R)$ is a dominating set of $G(R)$.

Theorem 3.10 ([6], Proposition 4.6) *Let R be a commutative ring. If R is reduced, then the unit graph $G(R[x])$ is a disconnected graph.*

The following result gives us information on the connectedness of unit graphs associated with matrix rings.

Theorem 3.11 ([6], Proposition 4.7) *Let R be a field with $|R| > 2$, and let $M_n(R)$, $n > 1$, be the ring of $n \times n$ matrices over R. Then the unit graph $G(M_n(R))$ is a connected graph.*

By applying Maschke's theorem and the Wedderburn–Artin theorem, the above theorem implies the following result.

Theorem 3.12 ([6], Corollary 4.8) *Let R be an Artinian semisimple ring, G be a finite group such that $|G|$ is a unit in R, and let RG be the group ring. Then the unit graph $G(RG)$ is a connected graph.*

3.5 Further Properties of Unit Graphs

A *k-edge coloring* of a graph G is an assignment of k *labels*, also called *colors*, to the edges of G such that every pair of distinct edges meeting at a common vertex are assigned two different colors. If G has a k-edge coloring, then G is said to be *k-edge colorable*. The *chromatic index* of G, denoted by $\chi'(G)$, is the smallest number k such that G is k-edge colorable. By Vizing's theorem, if G is a graph whose maximum vertex degree is Δ, then $\Delta \leq \chi'(G) \leq \Delta + 1$. Vizing's theorem divides the graphs into two classes according to their chromatic index; graphs satisfying $\chi'(G) = \Delta$ are called *class* 1, those with $\chi'(G) = \Delta + 1$ are *class* 2.

Theorem 3.13 ([6], Theorem 5.2) *Let R be a finite ring. Then the unit graph $G(R)$ is class* 1.

One can easily see that, for example, $\operatorname{diam}(G(\mathbb{Z}_6)) = 3$, $\operatorname{diam}(G(\mathbb{Z}_8)) = 2$, and $\operatorname{diam}(G(\mathbb{Z})) = \infty$. If R is a field with $|R| > 2$, and if $M_n(R)$, $n > 1$, is the ring of $n \times n$ matrices over R, then $\operatorname{diam}(G(M_n(R))) \leq 4$.

Theorem 3.14 ([6], Corollary 5.6 and Theorem 5.7) *Let R be a finite commutative ring. Then* $\operatorname{diam}(G(R)) \in \{1, 2, 3, \infty\}$. *Moreover, the following statements hold:*

(a) $\operatorname{diam}(G(R)) = 1$ *if and only if R is a field with* $\operatorname{Char}(R) = 2$.
(b) $\operatorname{diam}(G(R)) = 2$ *if and only if one of the following cases occurs:*
  (1) *R is a field with* $\operatorname{Char}(R) \neq 2$,
  (2) *R is not a field and R cannot have $\mathbb{Z}_2$ as a quotient,*
  (3) *R is a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$ and $R \ncong \mathbb{Z}_2$.*
(c) $\operatorname{diam}(G(R)) = 3$ *if and only if R has $\mathbb{Z}_2$ as a quotient and cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient and R is not a local ring.*
(d) $\operatorname{diam}(G(R)) = \infty$ *if and only if R has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.*

The finiteness of R in Theorem 3.14 is essential. For example, the unit graphs $G(\mathbb{Z})$ and $G(\mathbb{Z} \times \mathbb{Z})$ are both connected, and we have $\operatorname{diam}(G(\mathbb{Z})) = \operatorname{diam}(G(\mathbb{Z} \times \mathbb{Z})) = \infty$. Note that $\mathbb{Z}$ has $\mathbb{Z}_2$ as a quotient and cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient, while $\mathbb{Z} \times \mathbb{Z}$ has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.

Theorem 3.15 ([6], Proposition 5.10) *Let R be a finite commutative ring. Then* $\operatorname{gr}(G(R)) \in \{3, 4, 6, \infty\}$.

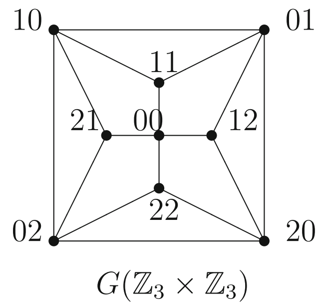

Fig. 4 A planar unit graph

Note that $\mathrm{gr}(G(\mathbb{Z}_5)) = 3$, $\mathrm{gr}(G(\mathbb{Z}_4)) = 4$, $\mathrm{gr}(G(\mathbb{Z}_6)) = 6$, and $\mathrm{gr}(G(\mathbb{Z}_2)) = \infty$. Therefore, for every $n \in \{3, 4, 6, \infty\}$, there exists a ring R such that $\mathrm{gr}(G(R)) = n$. Suppose that R is a finite commutative ring, and M is a finite unitary R-module. Suppose that $R \oplus M$ denotes the Nagata extension of R by M. If $|M| > 2$, then $\mathrm{gr}(G(R \oplus M)) \leq 4$. In particular, if $2 \in U(R)$, then $\mathrm{gr}(G(R \oplus M)) = 3$.

Unit graphs are not planar, in general. An example to illustrate this fact is the unit graph $G(\mathbb{Z}_{10})$, which is not planar. On the other hand, there are also planar unit graphs. For example, the unit graph $G(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a planar graph (cf. Fig. 4).

The following theorem gives necessary and sufficient condition for the planarity of $G(R)$. Here, $\mathbb{F}_4$ denotes the field with four elements.

Theorem 3.16 ([6], Theorem 5.14) *Let R be a finite commutative ring. Then the unit graph $G(R)$ is planar if and only if R is isomorphic to one of the following rings:*

(a) $\mathbb{Z}_5$,

(b) $\mathbb{Z}_3 \times \mathbb{Z}_3$,

(c) $\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell \text{ times}} \times S$, *where* $\ell \geq 0$ *and* $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$, $S \cong \mathbb{Z}_4$, $S \cong \mathbb{F}_4$, *or* $S \cong \left\{ \left[\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_2 \right\}$.

The following theorem concerns the weak perfectness of unit graphs.

Theorem 3.17 ([22], Theorem 2.2) *If R is a ring, then the unit graph $G(R)$ is weakly perfect.*

Let us note that the proof of Theorem 3.17 gives us an explicit formula for the chromatic number of $G(R)$. For a given ring R, we may write $R \cong R_1 \times \cdots \times R_n$, where every R_i is a local ring with maximal ideal $\mathfrak{m}_i$. If $2 \notin U(R)$, by reordering, we may assume that $R_1, \ldots, R_\ell$ all have characteristic equal to 2 with $|R_1|/|\mathfrak{m}_1| \leq \cdots \leq |R_\ell|/|\mathfrak{m}_\ell|$ and $R_{\ell+1}, \ldots, R_n$ all have characteristic not equal to 2. Then,

$$\chi(G(R)) = \begin{cases} \frac{1}{2^n} \prod_{i=1}^{n} (|R_i| - |\mathfrak{m}_i|) + n & \text{if } 2 \in U(R), \\ |R_1|/|\mathfrak{m}_1| & \text{if } 2 \notin U(R). \end{cases}$$

Note that this formula reduces to the following simpler form if all of the R_is are fields.

$$\chi(G(R)) = \begin{cases} \frac{1}{2^n} \prod_{i=1}^{n} (|R_i| - 1) + n & \text{if } 2 \in U(R), \\ |R_1| & \text{if } 2 \notin U(R). \end{cases}$$

We conclude this section with two examples which illustrate Theorem 3.17 and the above formulas. Let $k > 1$ be an integer and write $k = p_1^{r_1} \ldots p_n^{r_n}$, where the p_is are distinct prime numbers and the r_is are positive integers. Therefore, we obtain $\mathbb{Z}_k \cong \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_n^{r_n}}$, and so we have the following formula, where φ denotes the Euler phi function:

$$\chi(G(\mathbb{Z}_k)) = \begin{cases} \frac{1}{2^n} \varphi(k) + n & \text{if } k \text{ is odd}, \\ 2 & \text{if } k \text{ is even}. \end{cases}$$

Therefore, for example, the chromatic number of $G(\mathbb{Z}_9)$ is equal to four and the chromatic number of $G(\mathbb{Z}_{12})$ is equal to two. In Fig. 5, we illustrate these points. Here, the different bullets indicate the presence of the different colors.

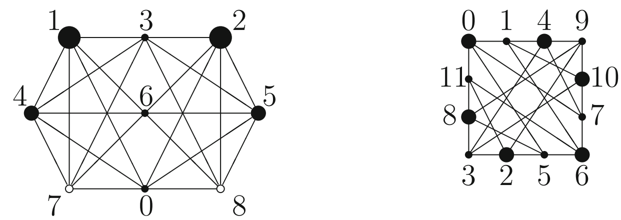

Fig. 5 The unit graphs $G(\mathbb{Z}_9)$ and $G(\mathbb{Z}_{12})$ and their four-coloring and two-coloring

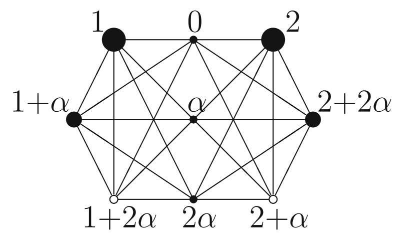

Fig. 6 The unit graph $G(\mathbb{Z}_3[x]/(x^2))$ and its four-coloring

The chromatic number of $G(\mathbb{Z}_3[x]/(x^2))$ is equal to four. In Fig. 6, we illustrate this point. Here, the different bullets indicate the presence of the different colors.

A graph is *Hamiltonian* if it has a cycle, which visits every vertex exactly once; such a cycle is called a *Hamiltonian cycle*. In general, the problem of finding a Hamiltonian cycle in a given graph is an NP-complete problem and it is a special case of the traveling salesman problem. This is a problem in combinatorial optimization studied in operations research and theoretical computer science (cf. [12]). The only known way to determine whether a given graph has a Hamiltonian cycle is to undertake an exhaustive search. Also, till now no theorem giving a necessary and sufficient condition for a graph to be Hamiltonian is known. The study of Hamiltonian graphs has long been an important topic.

Theorem 3.18 ([23], Theorem 2.1) *Let R be a ring such that $R \ncong \mathbb{Z}_2$ and $R \ncong \mathbb{Z}_3$. Then the following statements are equivalent:*

(a) *The unit graph $G(R)$ is Hamiltonian.*
(b) *The ring R cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.*
(c) *The ring R is generated by its units.*
(d) *The unit sum number of R is less than or equal to ω.*
(e) *The unit graph $G(R)$ is connected.*

4 Total Graphs

In 2008, Anderson and Badawi [4] introduced the notion of total graphs of rings.

4.1 Total Graphs: Definition and Remarks

Let R be a commutative ring and $Z(R)$ be the set of zero divisors of R. The *total graph* of R, denoted by $T(\Gamma(R))$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x + y \in Z(R)$.

Three important subgraphs of total graphs are defined below. Let R be a commutative ring with $Reg(R)$ its set of regular elements, $Z(R)$ its set of zero divisors, and $Nil(R)$ its ideal of nilpotent elements. Let $T(\Gamma(R))$ be the total graph of R. Then $Reg(\Gamma(R))$ is the subgraph of $T(\Gamma(R))$ with vertices $Reg(R)$, $Z(\Gamma(R))$ is the subgraph of $T(\Gamma(R))$ with vertices $Z(R)$, and $Nil(\Gamma(R))$ is the subgraph of $T(\Gamma(R))$ with vertices $Nil(R)$.

4.2 Basic Properties of Total Graphs

The study of $T(\Gamma(R))$ breaks naturally into two cases depending on whether or not $Z(R)$ is an ideal of R. In the following, we provide general structure theorems for $Reg(\Gamma(R))$, $Z(\Gamma(R))$, and $Nil(\Gamma(R))$ in the case where $Z(R)$ is an ideal of R.

Theorem 4.1 ([4], Theorem 2.2) *Let R be a commutative ring such that $Z(R)$ is an ideal of R, and let $|Z(R)| = \alpha$ and $|R/Z(R)| = \beta$.*

(1) *If $2 \in Z(R)$, then $Reg(\Gamma(R))$ is the union of $\beta - 1$ disjoint K_αs.*
(2) *If $2 \notin Z(R)$, then $Reg(\Gamma(R))$ is the union of $(\beta - 1)/2$ disjoint $K_{\alpha,\alpha}$s.*

Theorem 4.2 ([4], Theorem 2.1) *Let R be a commutative ring such that $Z(R)$ is an ideal of R. Then $Z(\Gamma(R))$ is a complete $(induced)$ subgraph of $T(\Gamma(R))$ and $Z(\Gamma(R))$ is disjoint from $Reg(\Gamma(R))$.*

Theorem 4.3 ([4], Theorem 2.10) *Let R be a commutative ring:*

(1) *$Nil(\Gamma(R))$ is a complete $(induced)$ subgraph of $Z(\Gamma(R))$.*
(2) *Each vertex of $Nil(\Gamma(R))$ is adjacent to each distinct vertex.*
(3) *$Nil(\Gamma(R))$ is disjoint from $Reg(\Gamma(R))$.*
(4) *If $\{0\} \neq Nil(R) \subseteq Z(R)$, then $\mathrm{gr}(Z(\Gamma(R))) = 3$.*

4.3 Further Properties of Total Graphs

In [4], the authors have computed the diameter and the girth of $Reg(\Gamma(R))$, and show that $Z(\Gamma(R))$ is always connected and $T(\Gamma(R))$ is never connected if $Z(R)$ is an ideal. Also, when $Z(R)$ is not an ideal of R, they have shown that $Z(\Gamma(R))$ is always connected but never complete, and $Z(\Gamma(R))$ and $Reg(\Gamma(R))$ are never disjoint subgraphs of $T(\Gamma(R))$ and $|Z(R)| > 3$.

In the following, further properties of $T(\Gamma(R))$ are given.

Theorem 4.4 ([25], Lemma 1.1) *Let R be a commutative ring. Let x be a vertex of $T(\Gamma(R))$. Then the degree of x is either $|Z(R)|$ or $|Z(R)| - 1$. In particular, if $2 \in Z(R)$, then $T(\Gamma(R))$ is a $(|Z(R)| - 1)$-regular graph.*

Let S_k denote the sphere with k handles, where k is a nonnegative integer, that is, S_k is an oriented surface of genus k. The *genus* of a graph G, denoted by $\gamma(G)$, is the minimum integer n such that the graph can be embedded in S_n. Therefore, a graph G is planar if and only $\gamma(G) = 0$. A graph is called *toroidal* if $\gamma(G) = 1$. Note that if H is a subgraph of a graph G, then $\gamma(H) \leq \gamma(G)$.

Theorem 4.5 ([25], Lemma 1.3) *Let $\mathbb{F}_q$ denote the field with q elements. Then the total graph of $\mathbb{F}_2 \times \mathbb{F}_q$ is isomorphic to $K_2 \times K_q$. Furthermore, for any positive integer m and $q > 2$,*

$$\gamma(T(\Gamma(\mathbb{F}_{2^m} \times \mathbb{F}_q))) \geq 2^m \left\lceil \frac{(q-3)(q-4)}{12} \right\rceil.$$

Theorem 4.6 ([25], Theorem 1.4) *For any positive integer g, there are finitely many finite rings R whose total graph has genus g.*

Theorem 4.7 ([25], Theorem 1.5) *Let R be a finite commutative ring such that $T(\Gamma(R))$ is planar. Then the following hold:*

(a) *If R is a local ring, then R is a field or R is isomorphic to the one of the 9 following rings: $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2[x, y]/(x, y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_8$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$.*
(b) *If R is not a local ring, then R is an infinite integral domain or R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$.*

Theorem 4.8 ([25], Theorem 1.6) *Let R be a finite commutative ring such that $T(\Gamma(R))$ is toroidal. Then the following statements hold:*

(a) *If R is a local ring, then R is isomorphic to $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$.*
(b) *If R is not a local ring, then R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{F}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ (Fig. 7).*

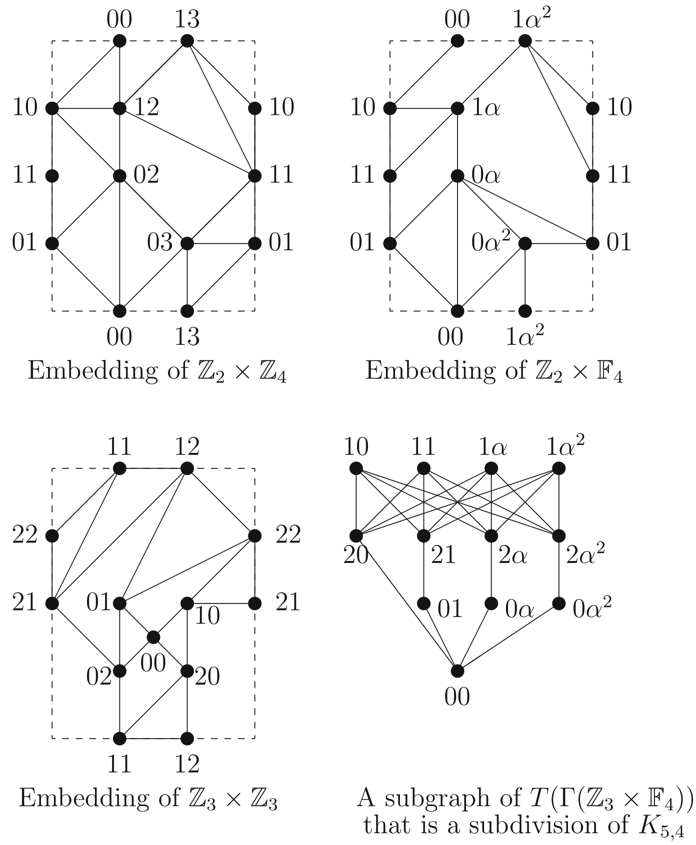

Fig. 7 Embeddings in the torus and a subgraph of $T(\Gamma(\mathbb{Z}_3 \times \mathbb{F}_4))$

5 Graphs Similar to Known Graphs

Let R be a ring and $U(R)$ be the set of unit elements of R. The *counit graph* of R, denoted by $\widetilde{G}(R)$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x - y \in U(R)$. This graph was defined in a paper of Lucchini and Maróti [19]. It is very similar to the unit graph. For this graph, one can deduce similar properties such as those of unit graphs. In some cases, dealing with this graph is better than with the unit ones.

In this direction, we may define a graph similar to the total graph. In the following, we define this graph, and then we give an application of it to the rings.

Definition 5.1 Let R be a ring and $Z(R)$ be the set of zero divisors of R. The *cototal graph* of R, denoted by $\widetilde{T}(\Gamma(R))$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x - y \in Z(R)$.

By using cototal graphs, we characterize the rings which are additively generated by their zero divisors.

Theorem 5.2 *Let R be a finite ring. Then the following statements are equivalent:*

(a) *The ring R is generated by its zero divisors.*
(b) *The cototal graph $\widetilde{T}(\Gamma(R))$ is connected.*
(c) *The cototal graph $\widetilde{T}(\Gamma(R))$ is Hamiltonian.*

For the proof of Theorem 5.2, we need the following well-known fact. We state and prove it here for the convenience of the reader.

Lemma 5.3 *Let x and y be distinct vertices of a graph G. If there is a walk between x and y, then there is also a path between x and y.*

Proof By assumption, there is a walk between x and y, and so we may select a walk

$$W : x = v_0 \xrightarrow{e_1} v_1 \longrightarrow \cdots \xrightarrow{e_k} v_k = y$$

of minimal length k between x and y. If W is not a path, select a vertex that appears twice, say $v_i = v_j$ where $i < j$. Consider

$$W' : x = v_0 \xrightarrow{e_1} v_1 \longrightarrow \cdots \xrightarrow{e_i} v_i \xrightarrow{e_{j+1}} v_{j+1} \longrightarrow \cdots \xrightarrow{e_k} v_k = y.$$

Then W' is a walk between x and y with length shorter than k, a contradiction. Therefore, W is a path between x and y. □

Proof of Theorem 5.2. (a) ⇒ (b): Suppose that $a \in R$. Since R is generated by its zero divisors, we may write $a = c_1 + \cdots + c_n$, where $c_i \in Z(R)$, $1 \le i \le n$. We now have the walk

$$0 \xrightarrow{e_1} c_1 \xrightarrow{e_2} c_1 + c_2 \longrightarrow \cdots \xrightarrow{e_n} c_1 + \cdots + c_n = a$$

between 0 and a.

This implies that for every $x, y \in R$, there is a walk W_1 between x and 0, as well as a walk W_2 between 0 and y. The walks W_1 and W_2 together form a walk W between x and y. By using Lemma 5.3, we conclude that there is also a path P between x and y, which implies the connectedness of $\widetilde{T}(\Gamma(R))$.

(b) ⇒ (a): Suppose that $a \in R$ can be written as a sum of zero divisors and $b \in R$ is adjacent to a in $\widetilde{T}(\Gamma(R))$. Therefore $a - b \in Z(R)$, and so we may write $b = a - c$, for some $c \in Z(R)$. Thus, b is a sum of zero divisors.

Now suppose that $x \in R$ is given. Since $\widetilde{T}(\Gamma(R))$ is connected, there exists a path between x and 0 and, therefore, by the above observation, we conclude that x is a sum of zero divisors. This means that R is generated by its zero divisors. □

Let G_1 and G_2 be two graphs. An *isomorphism* from G_1 to G_2 is a bijection $f : V(G_1) \longrightarrow V(G_2)$ such that $\{x, y\} \in E(G_1)$ if and only if $\{f(x), f(y)\} \in E(G_2)$. An *automorphism* of a graph G is an isomorphism from G to G. The set of all automorphisms of the graph G forms a group, denoted by $\mathrm{Aut}(G)$, called the *automorphism group* of G. A subgroup A of $\mathrm{Aut}(G)$ is called *transitive* if for every $x, y \in V(G)$, there is $f \in \mathrm{Aut}(G)$ such that $y = f(x)$. In the following, we state a significant result by Lovász [17] regarding these notions.

Lemma 5.4 ([17], p. 89, Exercise 17) *Let G be a connected graph such that* $\mathrm{Aut}(G)$ *contains a commutative transitive subgroup. Then G is a Hamiltonian graph.*

Proof of Theorem 5.2. (b) ⇒ (c): Suppose that $a \in R$ and define $f_a : R \longrightarrow R$ by $f_a(x) = x + a$. It is obvious that f_a is a bijection and $\{x, y\} \in E(\widetilde{T}(\Gamma(R)))$ if and only $\{f_a(x), f_a(y)\} \in E(\widetilde{T}(\Gamma(R)))$. Therefore, $f_a \in \mathrm{Aut}(\widetilde{T}(\Gamma(R)))$. Now, it is easy to see that $\{f_a \mid a \in R\}$ is a commutative transitive subgroup of $\mathrm{Aut}(\widetilde{T}(\Gamma(R)))$. Therefore by Lemma 5.4, we conclude that $\widetilde{T}(\Gamma(R))$ is Hamiltonian.

(c) ⇒ (b): It is obvious by definition. □

6 A Brief Glance at the Graphs Attached to Other Algebraic Structures

In the last decade, many authors have studied zero-divisor graphs of rings or other graphs associated with other algebraic structures. For instance, Nimbhorkar et al. [30] have shown that Beck's conjecture holds true for commutative semigroups with zero in which each element is idempotent. These semigroups are called meet-semilattices. Also, there are many graphs, which are attached to the groups. Perhaps the best known one is the Cayley graph of a group. For other ones, we may point to the prime graph and the commutating graph of a finite group, and graphs associated with character degrees as well as conjugacy class sizes in finite groups. For a survey and recent results concerning the two last ones, we refer the reader to [16]. Recently, Halaš and Jukl [15] introduced the zero-divisor graphs of posets and answered Beck's conjecture affirmatively in this case. More recently, a different method of associating a zero-divisor graph to a poset P was proposed by Lu and Wu in [18]. The graph defined by them is slightly different from the one defined in [15] and [35], where the vertex set of the graph consists of all the elements of P. The vertex set of the graph defined in [18] consists of all nonzero zero divisors of P. The study of the zero-divisor graphs of posets was then continued by Xue and Liu in [35]. Let us we give a short review of zero-divisor graphs of posets.

Let P be a nonempty set. A binary relation $\leq$ on P is called a *partial order* on P if $\leq$ is reflexive, anti-symmetric and transitive. A set that is equipped with a partial order is called a *partially ordered set* (for short: *poset*). Let P be a poset and let Q be nonempty subset of P. If there exists $y \in Q$ such that $y \leq x$ for every $x \in Q$, then y is called the *minimum* element of Q. Now, let P be a poset with a minimum element 0. We denote $P\backslash\{0\}$ by $P^{\times}$. For every $x, y \in P$, denote $L(x, y) = \{z \in P \mid z \leq x \text{ and } z \leq y\}$. An element $x \in P$ is called a *zero-divisor* of P if there exists $y \in P^{\times}$ such that $L(x, y) = \{0\}$. We denote the set of zero divisors of P by $Z(P)$ and we consider $Z(P)^{\times} := Z(P)\backslash\{0\}$. The *zero-divisor graph* of P, denoted by $\Gamma(P)$, is the graph obtained by setting all the elements of $Z(P)^{\times}$ to be the vertices and defining distinct vertices x and y to be adjacent if and only if $L(x, y) = \{0\}$.

In [10], Chakrabarty et al. have considered the intersection graph $G(R)$ of nontrivial left ideals of a ring R, that is, a graph whose vertices are in a one-to-one correspondence with all nontrivial left ideals of R and two distinct vertices are joined by an edge if and only if the corresponding left ideals of R have a nontrivial intersection. They have characterized the rings R for which the graph $G(R)$ is connected and have obtained several necessary and sufficient conditions on the ring R such that $G(R)$ is complete. For a commutative ring R with identity, they showed that $G(R)$ is complete if and only if $G(R[x])$ is also complete. In particular, they determined the values of n for which $G(\mathbb{Z}_n)$ is connected, complete, bipartite, planar or has a cycle. They also characterized the finite graphs which arise as the intersection graphs of $\mathbb{Z}_n$ and determined the set of all nonisomorphic graphs of $\mathbb{Z}_n$ for a given number of vertices. Finally, they determined the values of n for which the graph of $\mathbb{Z}_n$ is Eulerian and Hamiltonian.

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