ON THE h-VECTOR OF A SIMPLICIAL COMPLEX WITH SERRE'S CONDITION

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Abstract. Let Δ be a $(d-1)$-dimensional simplicial complex and let $h(\Delta) = (h_0, h_1, \ldots, h_d)$ be its h-vector. A recent result of Murai and Terai guarantees that if Δ satisfies Serre's condition (S_r), then $(h_0, h_1, \ldots, h_r)$ is an M-vector and $h_r + h_{r+1} + \cdots + h_d$ is nonnegative. In this article, we extend the result of Murai and Terai by giving r extra necessary conditions. More precisely, we prove that if Δ satisfies Serre's condition (S_r), then $\binom{i}{i}h_r + \binom{i+1}{i}h_{r+1} + \cdots + \binom{i+d-r}{i}h_d$, $0 \le i \le r \le d$, are all nonnegative.

1. Introduction and preliminaries

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts. One of the fastest developing subfields within algebraic combinatorics is combinatorial commutative algebra. It has evolved into one of the most active areas of mathematics during the past several decades. We refer the reader to the book by Stanley [13] as a general reference in the subject.

The study of h-vectors of simplicial complexes has long been a topic of interest both in combinatorics and combinatorial commutative algebra (see, for example, [1, 12, 15, 3, 2, 17, 9, 14, 10, 11]). Let Δ be a $(d-1)$-dimensional simplicial complex and let $h(\Delta) = (h_0, h_1, \ldots, h_d)$ be its h-vector. One of the fundamental questions in the subject is when $h_0, \ldots, h_d$ are all nonnegative. A classical result of Stanley guarantees that if Δ is Cohen–Macaulay, then h_i is nonnegative for every i with $0 \le i \le d$ (see [13, Theorem 3.3, Page 59]).

Recently, Murai and Terai [10] have generalized the classical result of Stanley in terms of Serre's condition, which appear in commutative algebra. They have proved that if Δ satisfies Serre's condition (S_r), then $(h_0, h_1, \ldots, h_r)$ is an M-vector and $h_r + h_{r+1} + \cdots + h_d$ is nonnegative. It is natural to ask whether these conditions are sufficient for a simplicial complex to satisfy Serre's condition. In fact, this is not the case, even for the independence simplicial complex of cycle graphs. In Theorem 2.1, we extend the result of Murai and Terai by giving r extra necessary conditions. These conditions are in fact sufficient for the independence simplicial complex of cycle

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2000 *Mathematics Subject Classification.* Primary: 13F55, 05E45; Secondary: 13C14.

Key words and phrases. Simplicial complex, h-Vector, Ring with Serre's condition.

The research of M. R. Pournaki and S. Yassemi was in part supported by a grant from IPM (No. 89130119 and No. 89130213).

graphs and quasi-forest simplicial complexes to satisfy Serre's condition (see Remark 2.2 and Corollary 2.5).

For convenience of the reader, let us recall some notions. Let $S = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$ and let M be a nonzero finitely generated S-module. We say that M satisfies *Serre's condition* (S_r), or simply M is a (S_r) *module*, if for every $\mathfrak{p} \in \mathrm{Spec(S)}$, the inequality depth $M_{\mathfrak{p}} \geq \min\{r, \dim M_{\mathfrak{p}}\}$ holds true. It is easy to see that M is Cohen–Macaulay if and only if it is a (S_r) module for all $r \geq 1$. We say that a simplicial complex Δ satisfies *Serre's condition* (S_r), or simply Δ is a (S_r) *simplicial complex*, if its Stanley–Reisner ring satisfies Serre's condition (S_r). Since every simplicial complex satisfies Serre's condition (S_1), we assume that $r \geq 2$. It is well known that if Δ is a (S_r) simplicial complex, then Δ is pure (see [10, Lemma 2.6]).

Let Δ be a simplicial complex. A facet F of Δ is called a *leaf* if there is a facet $G \neq F$ of Δ, called a *branch* of F, in such a way that $H \cap F \subseteq G \cap F$ holds true for all facets H of Δ with $H \neq F$. A *quasi-forest simplicial complex* is a simplicial complex Δ, which enjoys an ordering $F_1, F_2, \dots, F_s$ of the facets of Δ, called a *leaf order*, in such a way that for every j with $1 \leq j \leq s$ the facet F_j is a leaf of the subcomplex $\langle F_1, \dots, F_j \rangle$. One can easily check that if Δ is a quasi-forest simplicial complex and $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ is its f-vector, then $f_{d-1} \leq n - d + 1$ and the equality holds if and only if Δ is Cohen–Macaulay.

2. The main result

In this section, we state and prove the main result of this article. Recently, Murai and Terai [10] have proved that if Δ is a $(d-1)$-dimensional (S_r) simplicial complex and $h(\Delta) = (h_0, h_1, \dots, h_d)$ is its h-vector, then the following two conditions hold:

- $(h_0, h_1, \dots, h_r)$ is an M-vector, and
- $h_r + h_{r+1} + \cdots + h_d$ is nonnegative.

Note that in the above, a sequence of integers $\mathfrak{h} = (h_0, h_1, \dots, h_r)$ is called an *M-vector* provided there exists a Cohen–Macaulay $(r-1)$-dimensional simplicial complex Γ with $h(\Gamma) = \mathfrak{h}$. Now, it is natural to ask whether the above two conditions are sufficient for a simplicial complex to be (S_r). This is not the case, even for the independence simplicial complex of cycle graphs, as C_9 shows. We now extend the result of Murai and Terai by giving r extra necessary conditions to obtain sufficient conditions for the independence simplicial complex of cycle graphs as well as for quasi-forest simplicial complexes to be (S_r) (see Remark 2.2 and Corollary 2.5). Note that condition (2) in Theorem 2.1 includes $r + 1$ statements and it gives us the second condition of Murai and Terai for $i = 0$.

Theorem 2.1. *Let Δ be a $(d-1)$-dimensional (S_r) simplicial complex and let $h(\Delta) = (h_0, h_1, \dots, h_d)$ be its h-vector. Then the following conditions hold:*

(1) $(h_0, h_1, \dots, h_r)$ *is an* M*-vector, and*

(2) $\binom{i}{i}h_r + \binom{i+1}{i}h_{r+1} + \cdots + \binom{i+d-r}{i}h_d$ *is nonnegative for every* i *with* $0 \le i \le r \le d$.

Proof. Condition (1) was proved in [10]. We prove condition (2) by using induction on i. If $i = 0$, then condition (2) reduces to

$$h_r + h_{r+1} + \cdots + h_d \ge 0,$$

which is true by Theorem 1.3 of [10]. Let $1 \le i \le r$ be an arbitrary integer. We now suppose that condition (2) is true for $i - 1$ and prove that it holds true for i.

In order to do this, note that if $d < r$, then there is nothing to prove. Therefore, we assume that $d \ge r$ and we use induction on d. If $d = r$, then we must show that h_r is nonnegative and this is true since Δ is Cohen–Macaulay (see [13, Theorem 3.3, Page 59]). We now suppose that $d > r$. By [15, Proposition 2.3], we have the following identity, where $h_i(\mathrm{lk}_\Delta(v))$, $0 \le i \le d-1$, denotes the ith component of the h-vector of the simplicial complex $\mathrm{lk}_\Delta(v)$ and Δ is assumed on $[n] = \{1, \dots, n\}$:

$$ih_i + (d-i+1)h_{i-1} = \sum_{v \in [n]} h_{i-1}(\mathrm{lk}_\Delta(v)).$$

Therefore, we have

$$\begin{aligned}
(r+1)h_{r+1} + (d-r)h_r &= \sum_{v \in [n]} h_r(\mathrm{lk}_\Delta(v)),\\
(r+2)h_{r+2} + (d-r-1)h_{r+1} &= \sum_{v \in [n]} h_{r+1}(\mathrm{lk}_\Delta(v)),\\
&\ \vdots\\
dh_d + h_{d-1} &= \sum_{v \in [n]} h_{d-1}(\mathrm{lk}_\Delta(v)).
\end{aligned}$$

Now, by multiplying the jth equation by $\binom{i+j-1}{i}$, $1 \le j \le d-r-1$, and then adding the resulting equations, we conclude that

$$(d-r)h_r + \big((i+1)d - i(r+1)\big)h_{r+1} + \cdots + d\binom{i+d-r-1}{i}h_d =$$
$$\sum_{j=0}^{d-r-1} \binom{i+j}{i} \sum_{v \in [n]} h_{r+j}(\mathrm{lk}_\Delta(v)).$$

Since, by Lemma 2.2 of [4], $\mathrm{lk}_\Delta(v)$ is a (S_r) simplicial complex for every vertex v of Δ, by the induction hypothesis on d,

$$\sum_{j=0}^{d-r-1} \binom{i+j}{i} \sum_{v\in[n]} h_{r+j}(\mathrm{lk}_\Delta(v)) \geq 0,$$

and therefore we have

$$(d-r)h_r + \big((i+1)d - i(r+1)\big)h_{r+1} + \cdots + d\binom{i+d-r-1}{i} h_d \geq 0.$$

On the other hand, by the induction hypothesis on i, we have

$$\binom{i-1}{i-1} h_r + \binom{i}{i-1} h_{r+1} + \cdots + \binom{i+d-r-1}{i-1} h_d \geq 0.$$

Multiplying the last inequality by $r-i$ and adding the result to the previous inequality and then dividing by $d-i$ implies that

$$\binom{i}{i} h_r + \binom{i+1}{i} h_{r+1} + \cdots + \binom{i+d-r}{i} h_d \geq 0.$$

Therefore, condition (2) holds true by induction. □

Let C_n be the cycle graph with n vertices ($n \geq 3$). It is known that the independence simplicial complex of C_n is Cohen–Macaulay if and only if $n = 3, 5$ (see [16, Corollary 4.7]). Also, the independence simplicial complex of C_n is (S_2) if and only if $n = 3, 5, 7$ (see [5, Proposition 1.6]). Similarly, the independence simplicial complex of C_n is (S_r) for $r \geq 3$ if and only if $n = 3, 5$ (see [4, Proposition 4.2]). These observations together with an easy computation imply the following remark.

Remark 2.2. Let $(h_0, h_1, \ldots, h_d)$ be the h-vector of the independence simplicial complex of C_n. Then C_n is Cohen–Macaulay if and only if $h_0, \ldots, h_d$ are all nonnegative. Also C_n is (S_r) for $r \geq 2$ if and only if the following conditions hold:

(1) $h_0, \ldots, h_{r-1}$ are all nonnegative,

(2) $h_r + h_{r+1} + \cdots + h_d$ is nonnegative, and

(3) $h_r + 2h_{r+1} + \cdots + (d-r+1)h_d$ is nonnegative.

Therefore, conditions (1) and (2) of Theorem 2.1 are equivalent to C_n being (S_r).

Recently, the concept of quasi-forest simplicial complex has been extensively studied in combinatorics and combinatorial commutative algebra (see, for example, [8, 18, 6]). In the following we show that nonnegativity of the components of the h-vector of a quasi-forest simplicial complex is sufficient for Cohen–Macaulayness of it.

Proposition 2.3. *Let Δ be a $(d-1)$-dimensional quasi-forest simplicial complex and let $h(\Delta) = (h_0, h_1, \dots, h_d)$ be its h-vector. Then Δ is Cohen–Macaulay if and only if for every i with $0 \leq i \leq d$ we have $h_i \geq 0$.*

Proof. If Δ is Cohen–Macaulay, then by [13, Theorem 3.3, Page 59], for every i with $0 \leq i \leq d$, we have $h_i \geq 0$.

Conversely, suppose that for every i with $0 \leq i \leq d$, we have $h_i \geq 0$. Let $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ be the f-vector of Δ and $h(\Delta) = (h_0, h_1, \dots, h_d)$ be the h-vector of Δ. Since Δ is a quasi-forest simplicial complex, $f_{d-1} \leq n - d + 1$. Therefore, we conclude that $h_2 + h_3 + \cdots + h_d \leq 0$ and so by assumption, for every i with $2 \leq i \leq d$, we have $h_i = 0$. Now Theorem B of [7] implies that Δ is Cohen–Macaulay. □

In [5], it is shown that for the independence simplicial complex of bipartite and chordal graphs, Cohen–Macaulayness is equivalent to the Serre's condition (S_2). In the following proposition, we show that the same result holds true for quasi-forest simplicial complexes.

Proposition 2.4. *Let Δ be a $(d-1)$-dimensional quasi-forest simplicial complex. Then Δ is Cohen–Macaulay if and only if Δ satisfies Serre's condition (S_2).*

Proof. If Δ is Cohen–Macaulay, then obviously Δ satisfies Serre's condition (S_2).

Conversely, suppose that Δ satisfies Serre's condition (S_2). Let $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ be the f-vector of Δ and $h(\Delta) = (h_0, h_1, \dots, h_d)$ be the h-vector of Δ. Suppose, on the contrary, that Δ is not Cohen–Macaulay. We have $f_{d-1} \leq n - d$ and so $h_0 + h_1 + (h_2 + \cdots + h_d) \leq n - d$. Thus $h_2 + \cdots + h_d$ is negative, a contradiction. Therefore Δ is Cohen–Macaulay. □

Corollary 2.5. *Let Δ be a $(d-1)$-dimensional quasi-forest simplicial complex and let $h(\Delta) = (h_0, h_1, \dots, h_d)$ be its h-vector. Then Δ satisfies Serre's condition (S_2) if and only if $h_2 + h_3 + \cdots + h_d$ is nonnegative.*

Proof. If Δ satisfies Serre's condition (S_2), then by Theorem 1.1 of [10], $h_2+h_3+\cdots+h_d$ is nonnegative.

Conversely, assume that $h_2 + h_3 + \cdots + h_d$ is nonnegative. Let $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ be the f-vector of Δ and $h(\Delta) = (h_0, h_1, \dots, h_d)$ be the h-vector of Δ. Suppose, on the contrary, that Δ does not satisfies Serre's condition (S_2). Then by Proposition 2.4, Δ is not Cohen–Macaulay and so $f_{d-1} \leq n - d$. Thus $h_2 + h_3 + \cdots + h_d < 0$, a contradiction. Therefore Δ satisfies Serre's condition (S_2). □

Although, as we mentioned earlier, the conditions of Murai and Terai in [10] are not sufficient for a simplicial complex to be (S_r), even for the independence simplicial complex of a cycle graph, our conditions (1) and (2) stated in Theorem 2.1 are sufficient for the independence simplicial complex of cycle graphs as well as for quasi-forest simplicial complexes to be (S_r) (see Remark 2.2 and Corollary 2.5). Now, it is natural to ask whether the converse of Theorem 2.1 is true in general. We close the article by stating the following question which is a converse to Theorem 2.1 in a standard sense.

Question 2.6. *Let d and r be integers with $d \geq r \geq 2$ and let $\mathfrak{h} = (h_0, h_1, \ldots, h_d)$ be the h-vector of a simplicial complex in such a way that the following conditions hold:*

(1) *$(h_0, h_1, \ldots, h_r)$ is an M-vector, and*

(2) *$\binom{i}{i} h_r + \binom{i+1}{i} h_{r+1} + \cdots + \binom{i+d-r}{i} h_d$ is nonnegative for every i with $0 \leq i \leq r \leq d$.*

Does there exist a $(d-1)$-dimensional (S_r) simplicial complex Δ with $h(\Delta) = \mathfrak{h}$?

Acknowledgments

The authors would like to thank Satoshi Murai and Naoki Terai for reading an earlier version of this article and for their useful comments.

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