ON THE DIAMETER AND GIRTH OF ZERO-DIVISOR GRAPHS OF POSETS

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Abstract. In this paper, we deal with zero-divisor graphs of posets. We prove that the diameter of such a graph is either 1, 2 or 3 while its girth is either 3, 4 or ∞. We also characterize zero-divisor graphs of posets in terms of their diameter and girth.

1. Introduction

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. Associating a graph to an algebraic structure is a research subject in this area and has attracted considerable attention. In fact, the research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other. In 1988, Beck [3] introduced the idea of a zero-divisor graph of a commutative ring R with identity. He defined $\Gamma_0(R)$ to be the graph whose vertices are elements of R and in which two vertices x and y are adjacent if and only if $xy = 0$. He was mostly concerned with coloring of $\Gamma_0(R)$. Let $\chi(R)$ and $\omega(R)$ denote the chromatic number and the clique number of $\Gamma_0(R)$, respectively. Beck conjectured that $\chi(R) = \omega(R)$. Such graphs are called weakly perfect graphs. This investigation of coloring of a commutative ring was then continued by Anderson and Naseer in [1]. They gave a counterexample for the above conjecture of Beck. A different method of associating a zero-divisor graph to a commutative ring R was proposed by Anderson and Livingston in [2]. They believed that this better illustrated the zero-divisor structure of the ring. They defined $\Gamma(R)$ to be the graph whose vertices are nonzero zero-divisors of R and in which two vertices x and y are adjacent if and only if $xy = 0$. This graph is defined slightly different from the graph introduced by Beck who took the set of vertices to be the whole of R.

In the past decade, many authors have studied zero-divisor graphs of rings or other graphs associated to the other algebraic structures. For instance, Nimbhorkar et al. [14] have shown that Beck's conjecture holds true for commutative semigroups

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with zero in which each element is idempotent. These semigroups are called meet-semilattices. Recently, Halaš and Länger [9] have introduced the zero-divisor graphs of qosets and they have shown that Beck's conjecture holds true for these graphs. Also, Halaš and Jukl [8] introduced the zero-divisor graphs of posets and answered affirmatively to the Beck's conjecture. The study of the zero-divisor graphs of posets was then continued by Xue and Liu in [15]. More recently, a different method of associating a zero-divisor graph to a poset P was proposed by Lu and Wu in [13]. The graph defined by them is slightly different from the one defined in [8, 15], where the vertex-set of the graph consists of all the elements of P. The vertex-set of the graph defined in [13] consists of all nonzero zero-divisors of P.

In this paper, we deal with zero-divisor graphs of posets based on the terminology of [13]. We prove that the diameter of such a graph is either 1, 2 or 3 while its girth is either 3, 4 or ∞. We also characterize zero-divisor graphs of posets in terms of their diameter and girth.

2. Preliminaries

In this section, for convenience of the reader and also for later use, we recall some definitions and notations concerning posets. For undefined terms and concepts the reader is referred to [5].

Let P be a nonempty set. A binary relation $\leq$ on P is called a *partial order* on P if $\leq$ is reflexive, antisymmetric, and transitive. For $x, y \in P$, we write $y < x$ if $y \leq x$ and $y \neq x$. A set that is equipped with a partial order is called a *partially ordered set* or a *poset*, briefly.

Let P be a poset and let Q be a nonempty subset of P. If there exists $y \in Q$ such that $y \leq x$ for every $x \in Q$, then y is called the *least* element of Q. The least element, if exists, is unique because of the antisymmetry of the partial order. An element $x \in Q$ is called a *minimal* element of Q if $y \in Q$ and $y \leq x$ implies that $y = x$. We denote the set of minimal elements of Q by $\mathrm{Min}(Q)$.

Let P be a poset with least element 0. We denote $P \setminus \{0\}$ by $P^\times$. For every $x, y \in P$, denote $L(x, y) = \{z \in P \mid z \leq x \text{ and } z \leq y\}$. An element $x \in P$ is called a *zero-divisor* of P if there exists $y \in P^\times$ such that $L(x, y) = \{0\}$. We denote the set of zero-divisors of P by $Z(P)$ and we consider $Z(P)^\times := Z(P) \setminus \{0\}$. By an *ideal* of P we mean a nonempty subset I of P such that $x \in I$ and $y \leq x$ implies that $y \in I$. We say that I is *proper* if $I \neq P$. For $x \in P$, consider $(x] := \{y \in P \mid y \leq x\}$. It is easy to see that $(x]$ is an ideal of P, which is called the *principal ideal* of P generated by x. For $x \in P$, the *annihilator* of x, denoted by $\mathrm{Ann}(x)$, is defined to be $\{y \in P \mid L(x, y) = \{0\}\}$. It is easy to see that $\mathrm{Ann}(x)$ is an ideal of P. A proper ideal $\mathfrak{p}$ of P is called a *prime ideal* of P if for every $x, y \in P$, $L(x, y) \subseteq \mathfrak{p}$ implies that either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. We say that a prime ideal $\mathfrak{p}$ of P is an *annihilator prime ideal* of P if there exists $x \in P$ such that $\mathfrak{p} = \mathrm{Ann}(x)$. We denote the set of all annihilator prime ideals of P by $\mathrm{Ann}(P)$. Concerning prime ideals and annihilators for posets we refer the reader to [6, 7]. The *zero-divisor graph* of P, denoted by $\Gamma(P)$, is the graph obtained by setting all the elements of $Z(P)^{\times}$ to be the vertices and defining distinct vertices x and y to be adjacent if and only if $L(x, y) = \{0\}$.

Throughout the paper by a poset P we mean a nontrivial poset with least element 0 and $Z(P)^{\times} \neq \emptyset$.

3. The diameter of $\Gamma(P)$

In this section, a characterizing result regarding diameter of $\Gamma(P)$ is obtained (cf. Theorem 3.3). We first recall some basic definitions and remarks from graph theory. For undefined terms and concepts the reader is referred to [4].

Definitions and Remarks 3.1. Let G be a graph and suppose x and y are two vertices of G. We recall that a *walk* between x and y is a sequence $x = v_0, e_1, v_1, \dots, e_k, v_k = y$ of vertices and edges of G, denoted by

$$x = v_0 \xrightarrow{e_1} v_1 \longrightarrow \cdots \xrightarrow{e_k} v_k = y,$$

such that for every i with $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. Also a *path* between x and y is a walk between x and y without repeated vertices. A *cycle* of a graph is a path such that the start and end vertices are the same. The number of edges in a walk (counting repeats), path or a cycle, is called its *length*. We refer to a cycle with k edges as a k*-cycle*. For a graph G and vertices x and y of G, the *distance* between x and y, denoted by $d(x, y)$, is the number of edges in a shortest path between x and y. If there is no any path between x and y, then we write $d(x, y) = \infty$. Also we recall that the largest distance among all distances between pairs of the vertices of a graph G is called the *diameter* of G and is denoted by $\mathrm{diam}(G)$. A graph G is called *connected* if for any vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected*.

We continue the paper by collecting some basic facts about posets and their zero-divisor graphs for later use in the following lemma.

Lemma 3.2. *Let P be a poset. Then the following assertions hold:*

(a) *If $x, y \in \mathrm{Min}(P^{\times})$ with $x \neq y$, then $L(x, y) = \{0\}$ and so x and y are adjacent in $\Gamma(P)$.*

(b) *If $x \in \mathrm{Min}(P^{\times})$ and $y \in P$ with $x \nleq y$, then $L(x, y) = \{0\}$ and so x and y are adjacent in $\Gamma(P)$.*

(c) *For $x \in P^{\times}$, $\mathrm{Ann}(x) \neq \{0\}$ if and only if $x \in Z(P)^{\times}$.*

(d) *If $x, y \in P$ with $x \leq y$, then for every $z \in P$ we have $L(x, z) \subseteq L(y, z)$ and so $\mathrm{Ann}(y) \subseteq \mathrm{Ann}(x)$.*

(e) *If $x, y, z \in P^{\times}$ are such that $x \leq y$ and y is adjacent to z in $\Gamma(P)$, then x is adjacent to z in $\Gamma(P)$.*

(f) *If $y \in Z(P)^{\times}$, then $(y] \setminus \{0\} \subseteq Z(P)^{\times}$ and for every $x \in (y] \setminus \{0\}$ with $x \neq y$ we have $d(x, y) = 2$.*

(g) $|Z(P)^\times| \geq 2$.

(h) $\mathrm{Min}(P^\times) \subseteq Z(P)^\times$.

Proof. Parts (a)–(d), and (g) follow from the definitions of $L(x,y)$, $\mathrm{Min}(P^\times)$, $\mathrm{Ann}(x)$, and $Z(P)^\times$. For (g) we also need to use the assumption that $Z(P)^\times \neq \emptyset$. Also (e) follows from (d), and (f) follows from (e) noting that every $y \in Z(P)^\times$ is adjacent to some $z \in P^\times$ in $\Gamma(P)$. Finally for proof of (h), note that if $\mathrm{Min}(P^\times) = \emptyset$, then we are done. Therefore, suppose $\mathrm{Min}(P^\times) \neq \emptyset$ and let $x \in \mathrm{Min}(P^\times)$ be arbitrary. Let $y \in Z(P)^\times$. Then using (e) if $x \leq y$, or using (b) if $x \nleq y$, we have $x \in Z(P)^\times$. This implies that $\mathrm{Min}(P^\times) \subseteq Z(P)^\times$. □

The main result of this section is given as follows.

Theorem 3.3. *Let P be a poset. Then the following assertions hold:*

(a) $\Gamma(P)$ *is a connected graph with* $\mathrm{diam}(\Gamma(P)) \in \{1,2,3\}$.

(b) $\mathrm{diam}(\Gamma(P)) = 1$ *if and only if* $Z(P)^\times = \mathrm{Min}(P^\times)$.

(c) $\mathrm{diam}(\Gamma(P)) = 2$ *if and only if* $Z(P)^\times \setminus \mathrm{Min}(P^\times) \neq \emptyset$ *and* $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) \neq \{0\}$ *for every* $x, y \in Z(P)^\times \setminus \mathrm{Min}(P^\times)$ *with* $L(x,y) \neq \{0\}$.

(d) $\mathrm{diam}(\Gamma(P)) = 3$ *if and only if* $Z(P)^\times \setminus \mathrm{Min}(P^\times) \neq \emptyset$ *and* $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) = \{0\}$ *for some* $x, y \in Z(P)^\times \setminus \mathrm{Min}(P^\times)$ *with* $L(x,y) \neq \{0\}$.

Proof. (a) This is a particular case of Theorem 2.4 of [10]. However, for the sake of completeness, we give here a direct proof. Let x and y be two distinct vertices of $\Gamma(P)$, that is, $x, y \in Z(P)^\times$, $x \neq y$. We claim that $d(x,y) \in \{1,2,3\}$. If $L(x,y) = \{0\}$, then x is adjacent to y and so $d(x,y) = 1$. Now suppose that $L(x,y) \neq \{0\}$. Since $x, y \in Z(P)^\times$, we may choose $u, v \in Z(P)^\times$ such that $L(x,u) = L(y,v) = \{0\}$. If $L(u,v) = \{0\}$, then there is a path $x \to u \to v \to y$ in $\Gamma(P)$, and so $d(x,y) \leq 3$. If $L(u,v) \neq \{0\}$, then choosing $z \in L(u,v)$, $z \neq 0$, part (d) of Lemma 3.2 implies that $L(x,z) \subseteq L(x,u) = \{0\}$ and $L(y,z) \subseteq L(y,v) = \{0\}$. Therefore, $L(x,z) = L(y,z) = \{0\}$ and so $d(x,y) = 2$. Thus, our claim has been established. Hence $\Gamma(P)$ is a connected graph with $\mathrm{diam}(\Gamma(P)) \in \{1,2,3\}$.

(b) First suppose that $\mathrm{diam}(\Gamma(P)) = 1$. In view of part (h) of Lemma 3.2, it enough to show that $Z(P)^\times \subseteq \mathrm{Min}(P^\times)$. Therefore, let $y \in Z(P)^\times$. If $y \notin \mathrm{Min}(P^\times)$, then there exists a point $x \in (y] \setminus \{0\}$ with $x \neq y$, and so, by part (f) of Lemma 3.2, $x \in Z(P)^\times$ and $d(x,y) = 2$, which is a contradiction to the assumption that $\mathrm{diam}(\Gamma(P)) = 1$. Therefore $y \in \mathrm{Min}(P^\times)$ and so $Z(P)^\times \subseteq \mathrm{Min}(P^\times)$.

Conversely, if $Z(P)^\times = \mathrm{Min}(P^\times)$, then it follows from parts (a) and (g) of Lemma 3.2 that $\mathrm{diam}(\Gamma(P)) = 1$.

(c) First suppose that $\mathrm{diam}(\Gamma(P)) = 2$. Then, there exist $u, v \in Z(P)^\times$, $u \neq v$, such that $d(u,v) = 2$. Therefore, by part (a) of Lemma 3.2, $\{u,v\} \notin \mathrm{Min}(P^\times)$, whence $Z(P)^\times \setminus \mathrm{Min}(P^\times) \neq \emptyset$. Now, let $x, y \in Z(P)^\times \setminus \mathrm{Min}(P^\times)$ such that $L(x,y) \neq \{0\}$, i.e., $d(x,y) > 1$. Since $\mathrm{diam}(\Gamma(P)) = 2$, we have $d(x,y) = 2$. Therefore, there exists

$z \in Z(P)^\times$ with $d(x,z) = d(y,z) = 1$, which means that $z \in \mathrm{Ann}(x) \cap \mathrm{Ann}(y)$, that is, $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) \neq \emptyset$.

Conversely, suppose that $Z(P)^\times \setminus \mathrm{Min}(P^\times) \neq \emptyset$ and $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) \neq \{0\}$ for every $x, y \in Z(P)^\times \setminus \mathrm{Min}(P^\times)$ with $L(x,y) \neq \{0\}$. Let $x \in Z(P)^\times \setminus \mathrm{Min}(P^\times)$. Then, there is a point $z \in Z(P)^\times$ such that $L(x,z) = \{0\}$, i.e., $d(x,z) = 1$. Also, there exists $u \in P^\times$ such that $u < x$. Therefore, by part (f) of Lemma 3.2, we have $d(x,u) = 2$. Now, let $y \in Z(P)^\times$. If $y \notin \mathrm{Min}(P^\times)$, then, using the hypothesis, we have either $d(x,y) = 1$, or there exists a point $z \in Z(P)^\times$ with $d(x,z) = d(y,z) = 1$ which means that $d(x,y) = 2$. On the other hand, if $y \in \mathrm{Min}(P^\times)$, then, using parts (b) and (f) of Lemma 3.2, $d(x,y) = 1$ or 2 according as $y \nleq x$ or $y \leq x$. Hence, in view of part (a) of Lemma 3.2, it follows that $\mathrm{diam}(\Gamma(P)) = 2$.

(d) In view of (a)–(c), there is nothing to prove. □

As an immediate consequence, we have the following corollary.

Corollary 3.4. *Let P be a poset with $Z(P)^\times \setminus \mathrm{Min}(P^\times) \neq \emptyset$. Then* $\mathrm{diam}(\Gamma(P)) = \max\{2, d(x,y)\}$*, where the maximum is taken over all $x, y \in Z(P)^\times \setminus \mathrm{Min}(P^\times)$ with $x \neq y$. In particular, if $Z(P)^\times \setminus \mathrm{Min}(P^\times)$ is a chain, then* $\mathrm{diam}(\Gamma(P)) = 2$.

4. The girth of $\Gamma(P)$

In this section, some characterizing results regarding the girth of $\Gamma(P)$ are obtained (cf. Theorem 4.2 and Propositions 4.4, 4.9 and 4.11). We begin with recalling some more basic definitions and remarks from graph theory.

Definitions and Remarks 4.1. For a graph G, the *girth* of G is the length of a shortest cycle in G and is denoted by $\mathrm{girth}(G)$. If G has no cycles, we define the girth of G to be infinite. A *bipartite* graph is one whose vertex-set is partitioned into two (not necessarily nonempty) disjoint subsets in such a way that the two end vertices for each edge lie in distinct partitions. Among bipartite graphs, a *complete bipartite* graph is one in which each vertex is joined to every vertex that is not in the same partition. The complete bipartite graph with exactly two partitions of size m and n is denoted by $K_{m,n}$. Graphs of the form $K_{1,n}$ are called *star graphs*. A *cycle graph* is a graph that consists of a single cycle. In 1916, Hungarian mathematician, Dénes König (1884–1944) deduced that a graph is bipartite if and only if it contains no cycle of odd length (cf. [11]). His celebrated textbook *Theorie der endlichen und unendlichen Graphen* (1936) was the first book to present graph theory as a subject in its own right (cf. [12]).

One of the main results of this section is the following.

Theorem 4.2. *Let P be a poset. Then the following assertions hold:*

(a) $\mathrm{girth}(\Gamma(P)) \in \{3, 4, \infty\}$.

(b) $\mathrm{girth}(\Gamma(P)) = \infty$ *if and only if* $\Gamma(P)$ *is a star graph.*

(c) $\mathrm{girth}(\Gamma(P)) = 4$ *if and only if* $\Gamma(P)$ *is a bipartite but not a star graph.*

(d) $\mathrm{girth}(\Gamma(P)) = 3$ *if and only if* $\Gamma(P)$ *contains an odd cycle.*

Proof. (a) Suppose that $\mathrm{girth}(\Gamma(P)) \neq \infty$. Therefore, there exists a cycle of minimal length n in $\Gamma(P)$, say $x_1 \to x_2 \to x_3 \to \cdots \to x_n \to x_1$. Let $n \geq 5$. The minimality of n ensures that $L(x_2, x_4) \neq \{0\}$. Let $z \in L(x_2, x_4)$, $z \neq 0$. Then, by part (d) of Lemma 3.2, we have $L(x_1, z) \subseteq L(x_1, x_2) = \{0\}$ and $L(x_5, z) \subseteq L(x_5, x_4) = \{0\}$. It follows that $x_1 \to z \to x_5 \to \cdots \to x_n \to x_1$ is a cycle of length $n-2$ in $\Gamma(P)$. This contradicts the minimality of n. Therefore, we have $n = 3$ or 4, which implies that $\mathrm{girth}(\Gamma(P)) = 3$ or 4. All in all we obtain that $\mathrm{girth}(\Gamma(P)) \in \{3, 4, \infty\}$.

(b) First suppose that $\mathrm{girth}(\Gamma(P)) = \infty$. Let us assume that $\Gamma(P)$ is not a star graph. In this case we have $|Z(P)^{\times}| \geq 3$. Since, by Theorem 3.3, $\Gamma(P)$ is connected, there exists a point $x \in Z(P)^{\times}$ which is not an end vertex, that is, a vertex of the graph that has exactly one edge incident to. Since $\Gamma(P)$ is not a star graph, there exists a path of the form $a \to x \to b \to c$ in $\Gamma(P)$, where $a, b, c \in Z(P)^{\times}$. If a is adjacent to c, then $a \to x \to b \to c \to a$ is a cycle in $\Gamma(P)$, a contradiction. If a is not adjacent to c, then there exists $z \in P^{\times}$ such that $z \leq a$ and $z \leq c$, and so, by part (e) of Lemma 3.2, z is adjacent to both x and b, whence we have a cycle $z \to x \to b \to z$ in $\Gamma(P)$, a contradiction. Therefore $\Gamma(P)$ is a star graph.

Conversely, if $\Gamma(P)$ is a star graph, then it is easy to see that $\mathrm{girth}(\Gamma(P)) = \infty$.

(c) First assume that $\mathrm{girth}(\Gamma(P)) = 4$. Clearly, $\Gamma(P)$ is not a star graph. We show that $\Gamma(P)$ has no odd cycle. Then, in view of the well-known result of König mentioned earlier, $\Gamma(P)$ is bipartite. On the contrary, let us assume that $\Gamma(P)$ has an odd cycle and let $x_1 \to x_2 \to x_3 \to \cdots \to x_n \to x_1$ be an odd cycle of minimal length n in $\Gamma(P)$. Clearly $n \geq 5$, since $\mathrm{girth}(\Gamma(P)) \neq 3$. Now, the minimality of n ensures that $L(x_2, x_4) \neq \{0\}$. Let $z \in L(x_2, x_4)$, $z \neq 0$. Then, by part (d) of Lemma 3.2, we have $L(x_1, z) \subseteq L(x_1, x_2) = \{0\}$ and $L(x_5, z) \subseteq L(x_5, x_4) = \{0\}$. It follows that $x_1 \to z \to x_5 \to \cdots \to x_n \to x_1$ is an odd cycle of length $n-2$ in $\Gamma(P)$. This contradicts the minimality of n. Hence, $\Gamma(P)$ has no odd cycle. Alternatively, if $\mathrm{girth}(\Gamma(P)) = 4$, then the clique number of $\Gamma(P)$ is two. Therefore, by Theorem 2.9 of [8], the chromatic number of $\Gamma(P)$ is also two, which implies that $\Gamma(P)$ is bipartite.

Conversely, suppose that $\Gamma(P)$ is a bipartite but not a star graph. Then, once again appealing to the above-mentioned well-known result of König, we have $\mathrm{girth}(\Gamma(P)) \neq 3$. Also, by (b), $\mathrm{girth}(\Gamma(P)) \neq \infty$. Hence, we have $\mathrm{girth}(\Gamma(P)) = 4$.

(d) In view of (a)–(c) and the well-known result of König, there is nothing to prove. □

It is well known that if a graph G contains a cycle, then $\mathrm{girth}(G) \leq 2\mathrm{diam}(G) + 1$. Therefore, in view of Theorem 3.3, we may conclude, as in Theorem 2.4 of [10], that if $\Gamma(P)$ contains a cycle, then $\mathrm{girth}(\Gamma(P)) \leq 7$. In this context, Theorem 4.2 gives us a better upper bound.

Remark 4.3. Let P be a poset. If $\mathrm{Min}(P^{\times}) = \emptyset$, then clearly $\Gamma(P)$ is not a star graph, and so, $\mathrm{girth}(\Gamma(P)) = 3$ or 4.

We continue, as follows, to study the structure of the zero-divisor graph of a poset in terms of its girth.

Proposition 4.4. *Let P be a poset. Then* $\mathrm{girth}(\Gamma(P)) = 3$ *if one of the following conditions hold:*

(a) $|\mathrm{Min}(P^{\times})| \geq 3$.

(b) $|\mathrm{Ann}(P)| \geq 3$.

Proof. If part (a) holds, then by part (a) of Lemma 3.2, we have $\mathrm{girth}(\Gamma(P)) = 3$. If part (b) holds, then there exist $x, y, z \in P$ such that $\mathrm{Ann}(x)$, $\mathrm{Ann}(y)$, and $\mathrm{Ann}(z)$ are distinct prime ideals of P. Therefore, by Lemma 2.3 of [8], we have a cycle $x \to y \to z \to x$ in $\Gamma(P)$, and hence $\mathrm{girth}(\Gamma(P)) = 3$. □

In order to proceed further, we need the following lemmas of which the second one, namely, Lemma 4.6 is essentially a particular case of Lemma 2.12 of [10].

Lemma 4.5. *Let P be a poset and $x \in P^{\times}$. Then the following assertions are equivalent:*

(a) $x \in \mathrm{Min}(P^{\times})$.

(b) $\{0, x\}$ *is an ideal of P.*

Moreover, both the assertions hold if x is adjacent to an end vertex in $\Gamma(P)$. Consequently, $\Gamma(P)$ has no end vertex if $\mathrm{Min}(P^{\times}) = \emptyset$.

Proof. The equivalence of parts (a) and (b) follows from the definition of $\mathrm{Min}(P^{\times})$ and that of an ideal of P. The additional statement follows from part (e) of Lemma 3.2. □

Lemma 4.6. *Let P be a poset. If $a \to x \to b$ is a path in $\Gamma(P)$, then exactly one of the following assertions hold:*

(a) $\mathrm{Ann}(a) \cap \mathrm{Ann}(b) = \{0, x\}$ *and consequently* $x \in \mathrm{Min}(P^{\times})$.

(b) $a \to x \to b$ *is contained in a* 4*-cycle in* $\Gamma(P)$.

Proof. Since annihilators are ideals in P, the consequential part of (a) follows from Lemma 4.5. Next, it is easy to see that there exists a point $c \in \mathrm{Ann}(a) \cap \mathrm{Ann}(b)$ with $c \neq 0$ and $c \neq x$ if and only if there exists a 4-cycle of the form $a \to x \to b \to c \to a$ in $\Gamma(P)$. Hence, it follows that (a) does not hold if and only if (b) holds. This completes the proof. □

Note that the path $a \to x \to b$ in Lemma 4.6 may also be contained in a 3-cycle in $\Gamma(P)$.

Lemma 4.7. *Let P be a poset. If $a \to b \to c \to d$ is a path in $\Gamma(P)$ such that the edge $b \to c$ is not contained in a* 3*-cycle, then the vertices a and d are distinct and are adjacent to each other.*

Proof. Let $z \in L(a,d)$, $z \neq 0$. Then, by part (e) of Lemma 3.2, z is adjacent to both b and c, and hence, we have a 3-cycle $z \to b \to c \to z$ in $\Gamma(P)$. This contradiction proves the lemma. □

As an immediate consequence, we have the following corollary.

Corollary 4.8. *Let P be a poset. If $\Gamma(P)$ has no end vertex, then every edge in it is contained in a 3-cycle or a 4-cycle, and hence, $\Gamma(P)$ is a union of 3-cycles and 4-cycles.*

Along the same line, we also have the following results.

Proposition 4.9. *Let P be a poset such that $\mathrm{Min}(P^\times) = \emptyset$. Then every edge in $\Gamma(P)$ is contained in a 4-cycle, and hence, $\Gamma(P)$ is a union of 4-cycles.*

Proof. Consider an edge $a \to x$ in $\Gamma(P)$. By Lemma 4.5, since $a \notin \mathrm{Min}(P^\times)$, x is not an end vertex. Thus, there is a path $a \to x \to b$ in $\Gamma(P)$. Therefore, in view of Lemma 4.6 and the fact that $x \notin \mathrm{Min}(P^\times)$, the path $a \to x \to b$, and hence, the edge $a \to x$ is contained in a 4-cycle in $\Gamma(P)$. This proves the proposition. □

Note that Proposition 4.9 does not rule out the possibility that some edge in $\Gamma(P)$ is also contained in a 3-cycle, and that $\Gamma(P)$ is also a union of 3-cycles and 4-cycles.

Proposition 4.10. *Let P be a poset. If $\Gamma(P)$ has no end vertex, then every pair of vertices in $\Gamma(P)$ lie on a k-cycle, where $k \leq 6$.*

Proof. Let x and y be two distinct vertices in $\Gamma(P)$. Since $\mathrm{diam}(\Gamma(P)) \leq 3$, we have $d(x,y) = 1$, 2 or 3.

If $d(x,y) = 1$, then we have an edge $x \to y$ in $\Gamma(P)$. Therefore, by Corollary 4.8, x and y lie on a 3-cycle or a 4-cycle in $\Gamma(P)$.

If $d(x,y) = 2$, then there exists a path of the form $x \to w \to y$ in $\Gamma(P)$. Since $\Gamma(P)$ has no end vertex, there exist edges $x \to a$ and $y \to b$ in $\Gamma(P)$ with $a \neq w$ and $b \neq w$. Since $d(x,y) = 2$, we have, in view of part (e) of Lemma 3.2, $y \nleq a$ and $x \nleq b$. If $L(x,b) = \{0\}$, then we have a 4-cycle $b \to x \to w \to y \to b$ in $\Gamma(P)$ containing x and y. On the other hand, if $0 \neq z \in L(x,b)$, then $z \notin \{x,a,y,w\}$ and, in view of part (e) of Lemma 3.2, there exist edges $a \to z$ and $y \to z$ in $\Gamma(P)$, which means that we have a 5-cycle $a \to x \to w \to y \to z \to a$ in $\Gamma(P)$ containing x and y.

If $d(x,y) = 3$, then there exists a path of the form $x \to u \to v \to y$ in $\Gamma(P)$. Since $\Gamma(P)$ has no end vertex, there exist edges $x \to a$ and $y \to b$ in $\Gamma(P)$ with $a \neq u$ and $b \neq v$. Since $d(x,y) = 3$, we have $a \neq y, v, b$, and $b \neq x, u$. If $z \in L(u,b)$, $z \neq 0$, then, in view of part (e) of Lemma 3.2, there exist edges $x \to z$ and $y \to z$, contradicting the fact that $d(x,y) = 3$. Therefore, $L(u,b) = \{0\}$. Similarly, $L(a,v) = \{0\}$. Hence, we have a 6-cycle $a \to x \to u \to b \to y \to v \to a$ in $\Gamma(P)$ containing x and y. This completes the proof. □

One can observe that Corollary 4.8, and Propositions 4.9 and 4.10 are, in some sense, related to Theorem 2.13 of [10].

Finally, in this section, we have the following result.

Proposition 4.11. *Let P be a poset. If* $\mathrm{girth}(\Gamma(P)) = 4$*, then there is no end vertex in* $\Gamma(P)$*.*

Proof. Let a be an end vertex in $\Gamma(P)$. Choose a vertex b in $\Gamma(P)$ such that b is adjacent to a. By Theorem 4.2, $\Gamma(P)$ is not a star graph as $\mathrm{girth}(\Gamma(P)) < \infty$. Also, by Theorem 3.3, $\Gamma(P)$ is connected. Therefore, there is a path $a \to b \to c \to d$ in $\Gamma(P)$. Since $\mathrm{girth}(\Gamma(P)) = 4$, the edge $b \to c$ is not contained in a 3-cycle, and so, by Lemma 4.7, the vertices a and d are distinct and are adjacent to each other. Thus, a is adjacent to both b and d (note that $b \neq d$). This contradiction proves the proposition. □

In view of Corollary 4.8 and Proposition 4.10, it follows from Proposition 4.11 that if $\mathrm{girth}(\Gamma(P)) = 4$, then every edge in $\Gamma(P)$ is contained in a 3-cycle or a 4-cycle, and hence, $\Gamma(P)$ is a union of 3-cycles and 4-cycles, and every pair of vertices in $\Gamma(P)$ lie on a k-cycle, where $k \leq 6$.

We conclude our discussion with the following remark.

Remark 4.12. Let P be a poset. By Theorem 3.3, we have $\mathrm{diam}(\Gamma(P)) \in \{1, 2, 3\}$ and by Theorem 4.2 we have $\mathrm{girth}(\Gamma(P)) \in \{3, 4, \infty\}$. Therefore, it follows that

$$(\mathrm{diam}(\Gamma(P)), \mathrm{girth}(\Gamma(P))) \in \{(i, j) \mid \ 1 \leq i \leq 3 \text{ and } j = 3,\ 4 \text{ or } \infty\}.$$

Moreover, the following statements are valid:

(1) There is no poset P with $(\mathrm{diam}(\Gamma(P)), \mathrm{girth}(\Gamma(P))) = (1, 4),\ (3, 4)$ or $(3, \infty)$.
(2) $(\mathrm{diam}(\Gamma(P)), \mathrm{girth}(\Gamma(P))) = (1, 3)$ if and only if $|Z(P)^{\times}| \geq 3$ and $Z(P)^{\times} = \mathrm{Min}(P^{\times})$.
(3) $(\mathrm{diam}(\Gamma(P)), \mathrm{girth}(\Gamma(P))) = (1, \infty)$ if and only if $|Z(P)^{\times}| = 2$ and $Z(P)^{\times} = \mathrm{Min}(P^{\times})$.
(4) $(\mathrm{diam}(\Gamma(P)), \mathrm{girth}(\Gamma(P))) = (2, 4)$ if and only if $\Gamma(P)$ is a complete bipartite graph and $\Gamma(P)$ has a cycle.
(5) $(\mathrm{diam}(\Gamma(P)), \mathrm{girth}(\Gamma(P))) = (2, \infty)$ if and only if $\Gamma(P)$ is a star graph and $\Gamma(P) \neq K_2$, the complete graph with 2 vertices.
(6) There are several posets P with $(\mathrm{diam}(\Gamma(P)), \mathrm{girth}(\Gamma(P))) = (2, 3)$ or $(3, 3)$, but an elegant characterization for such posets is still eluding.

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