ON THE h-TRIANGLES OF SEQUENTIALLY (S_r) SIMPLICIAL COMPLEXES VIA ALGEBRAIC SHIFTING

M. R. POURNAKI, S. A. SEYED FAKHARI, AND S. YASSEMI

*Dedicated with gratitude to our teacher and friend
Jürgen Herzog on the occasion of his 70th birthday*

Abstract. Recently, Haghighi, Terai, Yassemi, and Zaare-Nahandi introduced the notion of a sequentially (S_r) simplicial complex. This notion gives a generalization of two properties for simplicial complexes: being sequentially Cohen–Macaulay and satisfying Serre's condition (S_r). Let Δ be a $(d-1)$-dimensional simplicial complex with $\Gamma(\Delta)$ as its algebraic shifting. Also let $(h_{i,j}(\Delta))_{0\leq j\leq i\leq d}$ be the h-triangle of Δ and $(h_{i,j}(\Gamma(\Delta)))_{0\leq j\leq i\leq d}$ be the h-triangle of $\Gamma(\Delta)$. In this paper, it is shown that for a Δ being sequentially (S_r) and for every i and j with $0 \leq j \leq i \leq r-1$, the equality $h_{i,j}(\Delta) = h_{i,j}(\Gamma(\Delta))$ holds true.

1. Introduction and preliminaries

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts. One of the fastest developing subfields within algebraic combinatorics is combinatorial commutative algebra. It has evolved into one of the most active areas of mathematics during the last several decades. Throughout the paper we deal with algebraic shifting and h-triangles of simplicial complexes. The study of these concepts has long been a topic of interest both in combinatorics and combinatorial commutative algebra. In this section, we recall some definitions and notation concerning combinatorics and combinatorial commutative algebra for later use. We refer the reader to the books by Stanley [13], Bruns and Herzog [2], Miller and Sturmfels [11], as well as Herzog and Hibi [8] as general references in the subject.

1.1. **Notions from combinatorics.** A *simplicial complex* Δ on the set of vertices $[n] := \{1,\ldots,n\}$ is a collection of subsets of $[n]$ which is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ, the *size* of a face F is defined to be $|F|$ and its *dimension* is defined to be $|F|-1$. (As usual, for a given finite set X, the number of elements of X is denoted by $|X|$.) The *dimension* of Δ which is denoted by $\dim\Delta$, is defined to be $d-1$, where

2000 *Mathematics Subject Classification.* Primary: 05E45, 13F55; Secondary: 13H10.

Key words and phrases. Simplicial complex, Simplicial complex with (S_r) property, Sequentially (S_r) simplicial complex, Algebraic shifting, h-Triangle.

The research of M. R. Pournaki and S. Yassemi was in part supported by a grant from IPM (No. 90130073 and No. 90130214).

$d = \max\{|F| \mid F \in \Delta\}$. The *degree* of a face $F \in \Delta$, denoted $\deg F$, is defined to be $\deg F = \max\{|G| \mid F \subseteq G \text{ and } G \in \Delta\}$. Also, the *degree* of Δ, denoted $\deg \Delta$, is defined to be $\deg \Delta = \min\{\deg F \mid F \in \Delta\}$. A *facet* of Δ is a maximal face of Δ with respect to inclusion. Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m \rangle$. We say that Δ is *pure* if all facets of Δ have the same cardinality. Note that Δ is pure if and only if all of its faces have the same degree. A *nonface* of Δ is a subset F of $[n]$ with $F \notin \Delta$. Let $\mathcal{N}(\Delta)$ denote the set of minimal nonfaces of Δ with respect to inclusion. The *link* of Δ with respect to a face $F \in \Delta$, denoted by $\mathrm{lk}_\Delta(F)$, is the simplicial complex $\mathrm{lk}_\Delta(F) = \{G \subseteq [n] \setminus F \mid G \cup F \in \Delta\}$. Let Δ and Δ' be two simplicial complexes on disjoint vertex sets V and W, respectively. The *join* $\Delta * \Delta'$ is the simplicial complex on the vertex set $V \cup W$ with faces $F \cup G$, where $F \in \Delta$ and $G \in \Delta'$.

Let Δ be a $(d-1)$-dimensional simplicial complex and let $-1 \leq i, j \leq d-1$. In 1996, Björner and Wachs (see [1, Definition 2.8]) defined $\Delta^{(i,j)}$ as

$$\Delta^{(i,j)} = \{F \in \Delta \mid \deg F \geq i+1 \text{ and } \dim F \leq j\}.$$

One can extend this definition to the case $i > d-1$, by defining $\Delta^{(i,j)}$ to be the empty simplicial complex. Clearly, $\Delta^{(i,j)}$ is a subcomplex of Δ. Throughout, we consider the following special cases: (i) the simplicial complex $\Delta^{(i)} := \Delta^{(-1,i)}$, called the ith *skeleton* of Δ; (ii) the simplicial complex $\Delta^{\langle i \rangle} := \Delta^{(i,d-1)}$, called ith *sequential layer* of Δ; (iii) the simplicial complex $\Delta^{[i]} := \Delta^{(i,i)}$, called the ith *pure skeleton* of Δ. Note that $\Delta^{\langle i \rangle}$ is the subcomplex of Δ generated by all facets whose dimension is at least i. That is, in fact, the subcomplex of all faces of Δ whose degree is at least $i+1$. Also, $\Delta^{[i]}$ is the pure subcomplex of Δ generated by all i-dimensional faces. The notation $\Delta^{[i]}$ is due to Wachs [14].

1.2. **Notions from combinatorial commutative algebra.** One of the connections between combinatorics and commutative algebra is via rings constructed from the combinatorial objects. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$ and let Δ be a simplicial complex on $[n]$. For every subset $F \subseteq [n]$, we set $x_F = \prod_{i \in F} x_i$. The *Stanley–Reisner ideal* of Δ is the ideal I_Δ of R which is generated by those squarefree monomials x_F with $F \notin \Delta$. In other words, $I_\Delta = \langle x_F \mid F \in \mathcal{N}(\Delta) \rangle$. The *Stanley–Reisner ring* of Δ, denoted by $\mathbb{K}[\Delta]$, is defined to be $\mathbb{K}[\Delta] = R/I_\Delta$.

We now recall two notions related to commutative algebra. Let M be a nonzero finitely generated R-module. We say that M is *Cohen–Macaulay*, if for every $\mathfrak{p} \in \mathrm{Spec(R)}$, the equality $\operatorname{depth} M_\mathfrak{p} = \dim M_\mathfrak{p}$ holds true. Also M is said to satisfy *Serre's condition* (S_r), or simply M is an (S_r) *module*, if for every $\mathfrak{p} \in \mathrm{Spec(R)}$, the inequality $\operatorname{depth} M_\mathfrak{p} \geq \min\{r, \dim M_\mathfrak{p}\}$ holds true. It is easy to see that M is Cohen–Macaulay if and only if it is an (S_r) module for all $r \geq 1$. We say that a simplicial complex Δ is *Cohen–Macaulay* if the Stanley–Reisner ring of Δ is Cohen–Macaulay. Also Δ is said to satisfy *Serre's condition* (S_r), or simply Δ is an (S_r) *simplicial complex*, if its Stanley–Reisner ring satisfies Serre's condition (S_r). Since every simplicial complex satisfies Serre's condition (S_1), we assume that $r \geq 2$. It is well known that if Δ is an (S_r) simplicial complex, then Δ is pure (see [12, Lemma 2.6]).

We now record the numerical data associated with a $(d-1)$-dimensional simplicial complex Δ. Let f_i denote the number of faces of Δ of dimension i. The sequence $f(\Delta) = (f_0, f_1, \dots, f_{d-1})$ is called the *f-vector* of Δ. Also to find out more about $f(\Delta)$, we need to study the h-vector of Δ, first defined by Stanley. Although it is seemingly complicated, but often is an elegant way to record the face numbers. Letting $f_{-1} = 1$ we define the *h-vector* $h(\Delta) = (h_0, h_1, \dots, h_d)$ of Δ by the formula

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

One can easily check that

$$f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j$$

and

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}.$$

It is well known that if $h(\Delta) = (h_0, h_1, \dots, h_d)$ is the h-vector of Δ and $H_{\mathbb{K}[\Delta]}(t)$ is the Hilbert series of $\mathbb{K}[\Delta]$, then we have

$$H_{\mathbb{K}[\Delta]}(t) = \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^d},$$

where $d = \dim \Delta + 1$ is the *Krull dimension* of $\mathbb{K}[\Delta]$.

2. Simplicial complexes satisfying the (d_r) property

Algebraic shifting is a procedure that to every simplicial complex Δ corresponds a new simplicial complex $\Gamma(\Delta)$, called the *shifted complex* of Δ, with the same h-vector as Δ and a nice combinatorial structure. Additionally, algebraic shifting preserves many algebraic and topological properties of the original complex, including Cohen–Macaulayness: a simplicial complex Δ is Cohen–Macaulay if and only if $\Gamma(\Delta)$ is Cohen–Macaulay, which, in turn, holds if and only if $\Gamma(\Delta)$ is pure (see Proposition 2.3).

Notation and Remarks 2.1. If $S = \{s_1 < \cdots < s_j\}$ and $T = \{t_1 < \cdots < t_j\}$ are two j-subsets of integers, then $S \leq_P T$ *under the standard partial order* if $s_p \leq t_p$ for all p. Also $S <_L T$ *under the lexicographic order* if there is a q such that $s_q < t_q$ and $s_p = t_p$ for all p with $p < q$. A collection $\mathcal{C}$ of two j-subsets S and T of integers is *shifted* if $S \leq_P T$ and $T \in \mathcal{C}$ together imply that $S \in \mathcal{C}$. A simplicial complex Δ is *shifted* if the set of j-dimensional faces of Δ is shifted for every j. In other words, a simplicial complex Δ on $[n]$ is shifted if, for $F \in \Delta$, $i \in F$ and $j \in [n]$ with $j < i$, one has $(F \setminus \{i\}) \cup \{j\} \in \Delta$.

Definitions and Remarks 2.2. Let Δ be a simplicial complex with the set of vertices $V = \{e_1, \ldots, e_n\}$ linearly ordered $e_1 < \cdots < e_n$. For a given field $\mathbb{K}$, let $\bigwedge \mathbb{K}V$ denote the *exterior algebra* of the vector space $\mathbb{K}V$; it has a $\mathbb{K}$-vector space basis consisting of all the monomials $e_S := e_{i_1} \wedge \cdots \wedge e_{i_j}$, where $S = \{e_{i_1} < \cdots < e_{i_j}\} \subseteq V$ and $e_\emptyset = 1$. Let $I(\Delta)$ be the ideal of $\bigwedge \mathbb{K}V$ generated by the set $\{e_S \mid S \notin \Delta\}$, and let $\widetilde{x}$ denote the image of $x \in \mathbb{K}V$ modulo $I(\Delta)$. Let $\{f_1, \ldots, f_n\}$ be a *generic basis* of $\mathbb{K}V$, i.e., $f_i = \sum_{j=1}^{n} \alpha_{i_j} e_j$, where the α_{i_j}'s are n^2 transcendentals, algebraically independent over $\mathbb{K}$. Define $f_S := f_{i_1} \wedge \cdots \wedge f_{i_k}$ for $S = \{i_1 < \cdots < i_k\}$ and set $f_\emptyset = 1$. Let

$$\Gamma(\Delta) := \big\{S \subseteq [n] \mid \widetilde{f_S} \notin \mathrm{Span}\{\widetilde{f_R} \mid R <_L S\}\big\}$$

be the *algebraically shifted complex* obtained from Δ. As the name implies, $\Gamma(\Delta)$ is a shifted simplicial complex, and it is independent of the numbering of the vertices of Δ or the choices of α_{i_j}.

As is often the case with algebraic shifting, we do not use the definition directly, but rather some theorems that characterize the results of algebraic shifting.

We continue this section by stating the following result. This result is just a translation of a purity result given by Kalai for algebraic shifting of Cohen–Macaulay simplicial complexes in terms of degree.

Proposition 2.3 (Kalai)**.** *Let Δ be a $(d-1)$-dimensional simplicial complex and let $\Gamma(\Delta)$ be its algebraic shifting. Then Δ is Cohen–Macaulay if and only if* $\deg \Gamma(\Delta) = d$.

Proof. It is known that Δ is Cohen–Macaulay if and only if $\Gamma(\Delta)$ is pure (see [10]). Therefore, Δ is Cohen–Macaulay if and only if $\Gamma(\Delta)$ has no facet of dimension less than $d-1$, which holds if and only if deg $\Gamma(\Delta) = d$, as requested. □

We now give the following definition to obtain an extension of Proposition 2.3.

Definition 2.4. Let $\mathbb{K}$ be a field, Δ be a simplicial complex and let $\mathbb{K}[\Delta]$ be its Stanley–Reisner ring. Then we say that Δ is a (d_r) *simplicial complex* (*over* $\mathbb{K}$), provided depth $\mathbb{K}[\Delta] \geq r$.

By using Exercise 5.1.23 of [2] and Corollary 4.5 of [3] we may conclude the following proposition.

Proposition 2.5. *Let Δ be a simplicial complex and let* $\Gamma(\Delta)$ *be its algebraic shifting. Then Δ is (d_r) if and only if* $\deg \Gamma(\Delta) \geq r$.

Remark 2.6. Note that Proposition 2.5 is a generalization of Proposition 2.3. In order to see this, let $\mathbb{K}$ be a field, Δ be a $(d-1)$-dimensional simplicial complex and $\mathbb{K}[\Delta]$ be its Stanley–Reisner ring. Then Δ is Cohen–Macaulay if and only if depth $\mathbb{K}[\Delta] \geq d$, which holds if and only if Δ is (d_d). Therefore, by Proposition 2.5,

Δ is Cohen–Macaulay if and only if deg $\Gamma(\Delta) \geq d$. But by the definition of degree, deg $\Gamma(\Delta)$ is always less than or equal to d, and therefore, we conclude that Δ is Cohen–Macaulay if and only if deg $\Gamma(\Delta) = d$, which is Proposition 2.3.

We now state and prove the following theorem which gives us a necessary and sufficient condition for a simplicial complex to be (d_r) in terms of reduced homology. For the definition of reduced homology we refer the reader to [2].

Theorem 2.7. *Let Δ be a $(d-1)$-dimensional simplicial complex and let $\mathbb{K}$ be a field. Then Δ is (d_r) if and only if $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);\mathbb{K}) = 0$ for all $\sigma \in \Delta$ with $|\sigma| \leq r$ and for all i with $i < \dim \mathrm{lk}_{\Delta^{(r-1)}}(\sigma)$.*

Proof. By using Exercise 5.1.23 of [2] we conclude that Δ is (d_r) if and only if $\Delta^{(r-1)}$ is Cohen–Macaulay. But Reisner's criterion (see [2, Corolarry 5.3.9]) implies that the latter is equivalent to the vanishing of $\widetilde{H}_i(\mathrm{lk}_{\Delta^{(r-1)}}(\sigma);\mathbb{K})$ for all $\sigma \in \Delta^{(r-1)}$ and all i with $i < \dim \mathrm{lk}_{\Delta^{(r-1)}}(\sigma)$. Since $\sigma \in \Delta^{(r-1)}$ is equivalent to $|\sigma| \leq r$, we obtain that Δ is (d_r) if and only if $\widetilde{H}_i(\mathrm{lk}_{\Delta^{(r-1)}}(\sigma);\mathbb{K}) = 0$ for all $\sigma \in \Delta$ with $|\sigma| \leq r$ and all i with $i < \dim \mathrm{lk}_{\Delta^{(r-1)}}(\sigma)$. But for every σ with $|\sigma| \leq r$ we have

$$\mathrm{lk}_{\Delta^{(r-1)}}(\sigma) = (\mathrm{lk}_\Delta(\sigma))^{(r-|\sigma|-1)},$$

and so for every i with $i < \dim \mathrm{lk}_{\Delta^{(r-1)}}(\sigma)$ we conclude that

$$\widetilde{H}_i(\mathrm{lk}_{\Delta^{(r-1)}}(\sigma);\mathbb{K}) = \widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);\mathbb{K}).$$

This implies that Δ is (d_r) if and only if $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma);\mathbb{K}) = 0$ for all $\sigma \in \Delta$ with $|\sigma| \leq r$ and all i with $i < \dim \mathrm{lk}_{\Delta^{(r-1)}}(\sigma)$, as requested. □

In the next proposition we collect some basic properties of (d_r) simplicial complexes.

Proposition 2.8. *Let Δ and Δ' be two simplicial complexes. Then the following statements hold:*

(1) *if Δ is (d_r), then $\Delta^{(i)}$ is (d_s) for all i, where $s = \min\{r, i+1\}$;*

(2) *if Δ is (d_r) and Δ' is $(d_{r'})$, then $\Delta * \Delta'$ is $(d_{r+r'})$;*

(3) *if Δ is (d_r), then $\mathrm{lk}_\Delta(v)$ is (d_{r-1}) for every vertex v of Δ.*

Proof. We easily conclude (1) by using [2, Exercise 5.1.23]. Also (2) is proved in [5, Lemmas 1 and 2]. For proving (3), we may assume that $r \geq 2$; otherwise, there is nothing to prove. Note that by Theorem 2.7, it is enough to prove that for a given field $\mathbb{K}$, $\widetilde{H}_i(\mathrm{lk}_{\mathrm{lk}_\Delta(v)}(\sigma);\mathbb{K}) = 0$ for all $\sigma \in \mathrm{lk}_\Delta(v)$ with $|\sigma| \leq r-1$ and all i with $i < \dim \mathrm{lk}_{(\mathrm{lk}_\Delta(v))^{(r-2)}}(\sigma)$. This is also equivalent to showing that for every $\sigma \in \mathrm{lk}_\Delta(v)$ with $|\sigma| \leq r-1$ and every i with $i < \dim \mathrm{lk}_{(\mathrm{lk}_\Delta(v))^{(r-2)}}(\sigma)$, $\widetilde{H}_i(\mathrm{lk}_\Delta(\sigma \cup \{v\});\mathbb{K}) = 0$. But $i < \dim \mathrm{lk}_{(\mathrm{lk}_\Delta(v))^{(r-2)}}(\sigma)$ implies that $i < \dim \mathrm{lk}_{\Delta^{(r-1)}}(\sigma \cup \{v\})$, and so Theorem 2.7 completes the proof (see also [12, Lemma 4.3] for another proof). □

We now state and prove the following lemma for later use.

Lemma 2.9. *Let Δ be an (S_r) simplicial complex and let $\Delta^{(i)}$ be its i-th skeleton. Then $\Delta^{(i)}$ is Cohen–Macaulay for every i with $1 \leq i \leq r-1$.*

Proof. Since Δ is an (S_r) simplicial complex, Proposition 2.3 of [7] implies that $\Delta^{(r-1)}$ is also (S_r). On the other hand, $\dim \Delta^{(r-1)} = r-1$. Therefore, we conclude that $\Delta^{(r-1)}$ is Cohen–Macaulay. For every i with $1 \leq i < r-1$, we have $\Delta^{(i)} = (\Delta^{(r-1)})^{(i)}$. Also, by Corollary 24.4 of [9], every skeleton of a Cohen–Macaulay simplicial complex is again Cohen–Macaulay. Hence, we conclude that $\Delta^{(i)}$ is Cohen–Macaulay. All in all, we obtain that for every i with $1 \leq i \leq r-1$, $\Delta^{(i)}$ is Cohen–Macaulay, as requested. $\square$

Let $\mathbb{K}$ be a field, Δ be a simplicial complex and let $\mathbb{K}[\Delta]$ be its Stanley–Reisner ring. By using Exercise 5.1.23 of [2] we conclude that

$$\operatorname{depth} \mathbb{K}[\Delta] = \max\left\{ i+1 \mid \Delta^{(i)} \text{ is Cohen} - \text{Macaulay} \right\}.$$

Therefore, Lemma 2.9 implies that if Δ is (S_r), then $\operatorname{depth} \mathbb{K}[\Delta] \geq r$ and so Δ is (d_r). This means that

$$(S_r) \quad \Longrightarrow \quad (d_r).$$

We now apply the above observation to one implication of Proposition 2.5 to get the following proposition, which in turn is another generalization of Proposition 2.3.

Proposition 2.10. *Let Δ be a simplicial complex and let $\Gamma(\Delta)$ be its algebraic shifting. If Δ is (S_r), then* $\deg \Gamma(\Delta) \geq r$.

Note that the converse of Proposition 2.10 is not true in general. In order to see this, let Δ be an (S_r) simplicial complex which is not Cohen–Macaulay. Then $\Gamma(\Delta)$ is not pure, while by Proposition 2.10 we have $\deg \Gamma(\Delta) \geq r$. Hence $\Gamma(\Gamma(\Delta)) = \Gamma(\Delta)$ has degree grater than or equal to r and is not (S_r).

Therefore it is natural to ask the following question.

Question 2.11. *What is the characterization of (S_r) simplicial complexes via algebraic shifting?*

3. On the h-triangle of a sequentially (S_r) simplicial complex

The h-vector of a simplicial complex plays an important role in the theory of Cohen–Macaulay complexes, including pure shellable complexes (see, for example, [13]). To extend part of this to the nonpure case we are led to introduce doubly indexed h-numbers. In order to do this, let Δ be a $(d-1)$-dimensional simplicial complex. For every i and j with $0 \leq j \leq i \leq d$, let $f_{i,j}$ denote the number of faces of Δ of degree i and size j and consider

$$h_{i,j} = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}.$$

Then the triangular integer arrays $\mathbf{f} = (f_{i,j})_{0 \leq j \leq i \leq d}$ and $\mathbf{h} = (h_{i,j})_{0 \leq j \leq i \leq d}$ are called the *f-triangle* and *h-triangle* of Δ, respectively. In order to show that $f_{i,j}$ (resp. $h_{i,j}$)

is the (i,j)th array of $\mathbf{f}$ (resp. $\mathbf{h}$), we sometimes write $f_{i,j}(\Delta)$ (resp. $h_{i,j}(\Delta)$) instead of $f_{i,j}$ (resp. $h_{i,j}$).

Before stating the results of this section, we define the notion of a sequentially (S_r) simplicial complex which was introduced in [7].

Definition 3.1. Let M be a finitely generated $\mathbb{Z}$-graded module over a standard graded $\mathbb{K}$-algebra R, where $\mathbb{K}$ is a field. For a positive integer r, we say that M is *sequentially* (S_r) if there exists a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ of M by graded submodules M_i satisfying the following two conditions:

(a) every quotient M_i/M_{i-1} satisfies the (S_r) condition of Serre;

(b) $\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_t/M_{t-1})$.

We say that a simplicial complex Δ on $[n] = \{1, \dots, n\}$ is *sequentially* (S_r) (*over a field* $\mathbb{K}$) if its Stanley–Reisner ring $\mathbb{K}[\Delta] = \mathbb{K}[x_1, \dots, x_n]/I_\Delta$, as a module over $R = \mathbb{K}[x_1, \dots, x_n]$ is sequentially (S_r).

Duval [3] has shown that algebraic shifting preserves the h-triangle of a simplicial complex Δ, provided Δ is sequentially Cohen–Macaulay. The analogue of Duval's result is given in the following theorem.

Theorem 3.2. *Let Δ be a sequentially (S_r) simplicial complex and let $\Gamma(\Delta)$ be its algebraic shifting. Then for every i and j with $0 \le j \le i \le r-1$, we have $h_{i,j}(\Delta) = h_{i,j}(\Gamma(\Delta))$.*

Proof. Let Δ be a sequentially (S_r) simplicial complex. By Theorem 2.6 of [7], the simplicial complex $\Delta^{[i]}$ for $i \le r-1$ is (S_r) and since its dimension is equal to i, we conclude that $\Delta^{[i]} = (\Delta^{\langle i \rangle})^{(i)}$ is Cohen–Macaulay. Now by Corollary 4.5 of [3], $\deg \Gamma(\Delta^{\langle i \rangle}) \ge i+1$ and so Theorem 4.6 of [3] implies that $\Gamma(\Delta^{\langle i \rangle}) = \Gamma(\Delta)^{\langle i \rangle}$. Therefore, for every j and for every i with $i \le r-1$, we have $f_j(\Gamma(\Delta^{\langle i \rangle})) = f_j(\Gamma(\Delta)^{\langle i \rangle})$, where f_j denotes the jth component of the f-vector. By using [3] we obtain that

$$f_{i,j}(\Delta) = f_{j-1}(\Delta^{\langle i-1 \rangle}) - f_{j-1}(\Delta^{\langle i \rangle}),$$

$$f_{i,j}(\Gamma(\Delta)) = f_{j-1}(\Gamma(\Delta)^{\langle i-1 \rangle}) - f_{j-1}(\Gamma(\Delta)^{\langle i \rangle}).$$

Therefore, for every i and j with $0 \le j \le i \le r-1$, we have $f_{i,j}(\Gamma(\Delta)) = f_{i,j}(\Delta)$. Now the relation between f-triangles and h-triangles completes the proof. □

We now state and prove our two next results. But first, let us to recall the following definition (see [1, Definition 2.1]).

Definition 3.3. A simplicial complex Δ is called *shellable* (in nonpure sense) if there exists a linear order $F_1, \dots, F_r$ of the facets of Δ such that for every i with $2 \le i \le r$, the simplicial complex $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$ is pure of dimension $\dim F_i - 1$.

Corollary 3.4. *Let Δ be a sequentially (S_r) simplicial complex. Then for every i and j with $0 \le j \le i \le r-1$, we have $h_{i,j}(\Delta) \ge 0$.*

Proof. By Theorem 3.2, for every i and j with $0 \leq j \leq i \leq r-1$, we have $h_{i,j}(\Delta) = h_{i,j}(\Gamma(\Delta))$. But shifted complexes are shellable and so by Theorem 3.4 of [1] their h-triangles are nonnegative. □

Corollary 3.5. *Let* $\mathbf{h} = (h_{i,j})_{0\leq j\leq i\leq d}$ *be an array of integers. Then the following conditions are equivalent:*

(1) *there exists a sequentially* (S_r) *simplicial complex* Δ *such that* $h_{i,j}(\Delta) = h_{i,j}$ *for every* i *and* j *with* $0 \leq j \leq i \leq r-1$;

(2) *there exists a shifted simplicial complex* Δ *such that* $h_{i,j}(\Delta) = h_{i,j}$ *for every* i *and* j *with* $0 \leq j \leq i \leq r-1$;

(3) *there exists a shellable simplicial complex* Δ *such that* $h_{i,j}(\Delta) = h_{i,j}$ *for every* i *and* j *with* $0 \leq j \leq i \leq r-1$;

(4) *there exists a sequentially Cohen–Macaulay simplicial complex* Δ *such that* $h_{i,j}(\Delta) = h_{i,j}$ *for every* i *and* j *with* $0 \leq j \leq i \leq r-1$.

Proof. (1)⇒(2) Consider the algebraic shifting of Δ and apply Theorem 3.2.

(2)⇒(3) Every shifted simplicial complex is shellable.

(3)⇒(4) Every shellable simplicial complex is sequentially Cohen–Macaulay.

(4)⇒(1) Every sequentially Cohen–Macaulay simplicial complex is sequentially (S_r). □

We now state and prove the following proposition. We recall that a sequence of integers $\mathfrak{h} = (h_0, h_1, \ldots, h_r)$ is an *M-vector* if there exists a Cohen–Macaulay $(r-1)$-dimensional simplicial complex Δ' with $h(\Delta') = \mathfrak{h}$.

Proposition 3.6. *Let* Δ *be a* $(d-1)$*-dimensional sequentially* (S_r) *simplicial complex. Then for every* c *with* $1 \leq c \leq r$, *the sequence* $(h'_{c,0}, h'_{c,1}, \ldots, h'_{c,c})$ *is an* M*-vector, where for every* j *with* $0 \leq j \leq c$,

$$h'_{c,j} = h_{c,j} + \sum_{s=c+1}^{d} \sum_{i=0}^{j} \binom{s-c-1+j-i}{j-i}.$$

Proof. By Theorem 2.6 of [7], for every c with $1 \leq c \leq r$ the simplicial complex $\Delta^{[c-1]} = \Delta^{(c-1,c-1)}$ is (S_r) and since its dimension is less than r, it is Cohen–Macaulay. Therefore its h-vector is an M-vector. Now Lemma 3.3(v) of [1] completes the proof. □

Proposition 3.7. *Let* Δ *be a* $(d-1)$*-dimensional sequentially* (S_r) *simplicial complex. Then for every* c *with* $r+1 \leq c \leq d$ *the following conditions hold, where for every* j *with* $0 \leq j \leq c$,

$$h'_{c,j} = h_{c,j} + \sum_{s=c+1}^{d} \sum_{i=0}^{j} \binom{s-c-1+j-i}{j-i}.$$

(1) *The sequence* $(h'_{c,0}, h'_{c,1}, \ldots, h'_{c,r})$ *is an* M*-vector*;

(2) *For every i with $0 \leq i \leq r$, we have*

$$\binom{i}{i}h'_{c,r} + \binom{i+1}{i}h'_{c,r+1} + \cdots + \binom{c+i-r}{i}h'_{c,c} \geq 0.$$

Proof. By Theorem 2.6 of [7], for every c with $r+1 \leq c \leq d$, the simplicial complex $\Delta^{[c-1]} = \Delta^{(c-1,c-1)}$ is (S_r) and by Lemma 3.3(v) of [1] the sequence

$$(h'_{c,0}, h'_{c,1}, \ldots, h'_{c,c})$$

is the h-vector of $\Delta^{(c-1,c-1)}$. Now, Theorem 3.1 of [6] completes the proof. $\square$

Iterated Betti numbers, introduced in [4], are a nonpure generalization of the reduced homology Betti numbers $\widetilde{\beta}_{i-1}(\Delta) = \dim_{\mathbb{K}} \widetilde{H}_{i-1}(\Delta)$. Although they can be defined as the Betti numbers of a certain chain complex (see [4, Section 4]), we will use the following equivalent formulation.

Notation and Remarks 3.8. Let Δ be a simplicial complex. For a set F of positive integers, let $\operatorname{init}(F) = \max\{r \mid \{1, \ldots, r\} \subseteq F\}$. Therefore, $\operatorname{init}(F)$ measures the largest "initial segment" in F, and is 0 if there is no initial segment, i.e., if 1 does not belong to F. Now, by Theorem 4.1 of [4], the ith iterated Betti number of Δ is

$$\beta_{j-1}[i](\Delta) = |\{F \in \mathcal{F}(\Gamma(\Delta)) \mid \dim F = j-1 \text{ and } \operatorname{init}(F) = i\}|.$$

A special case is $i = 0$. In this case, $\beta_j[0](\Delta) = \widetilde{\beta}_j(\Delta)$ is the ordinary Betti number of reduced homology. Björner and Wachs [1, Theorem 4.1] have shown that if Δ is nonpure and shellable, then

$$\widetilde{\beta}_{j-1}(\Delta) = h_{j,j}(\Delta)$$

for every j with $0 \leq j \leq d$. This equation was generalized in Theorem 1.2 of [4] to

$$\beta_{j-1}[i](\Delta) = h_{j,j-i}(\Delta)$$

for nonpure shellable Δ. This algebraic interpretation of the h-triangle of nonpure shellable complexes was part of the motivation for iterated Betti numbers. Theorem 3.2 allows us to generalize even further, by weakening the assumption on Δ from being nonpure shellable to being merely sequentially (S_r).

Corollary 3.9. *Let Δ be a sequentially (S_r) simplicial complex. Then for every i with $i \leq r-1$, we have $\beta_{j-1}[i](\Delta) = h_{j,j-i}(\Delta)$.*

Proof. By Theorem 5.4 of [4], $\beta_{j-1}[i](\Delta) = h_{j,j-i}(\Gamma(\Delta))$ for every simplicial complex Δ. We now apply Theorem 3.2 to obtain the result. $\square$

References

[1] A. Björner, M. L. Wachs, Shellable nonpure complexes and posets I, *Trans. Amer. Math. Soc.* **348** (1996), no. 4, 1299–1327.

[2] W. Bruns, J. Herzog, *Cohen–Macaulay Rings*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1993.

[3] A. M. Duval, Algebraic shifting and sequentially Cohen–Macaulay simplicial complexes, *Electron. J. Comb.* **3** (1996), no. 1, Research Paper 21, approx. 14 pp.

[4] A. M. Duval, L. L. Rose, Iterated homology of simplicial complexes, *J. Algebraic. Combin.* **12** (2000), no. 3, 279–294.
[5] R. Fröberg, A note on the Stanley–Reisner ring of a join and of a suspension, *Manuscripta Math.* **60** (1988), no. 1, 89–91.
[6] A. Goodarzi, M. R. Pournaki, S. A. Seyed Fakhari, S. Yassemi, On the h-vector of a simplicial complex with Serre's condition, *J. Pure Appl. Algebra* **216** (2012), no. 1, 91–94.
[7] H. Haghighi, N. Terai, S. Yassemi, R. Zaare-Nahandi, Sequentially S_r simplicial complexes and sequentially S_2 graphs, *Proc. Amer. Math. Soc.* **139** (2011), no. 6, 1993–2005.
[8] J. Herzog, T. Hibi, *Monomial Ideals*, Springer-Verlag, London, Ltd., London, 2011.
[9] T. Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carslaw Publicatin, 1992.
[10] G. Kalai, Algebraic shifting, *Adv. Stud. Pure Math.* **33** (2001), 121–163.
[11] E. Miller, B. Sturmfels, *Combinatorial Commutative Algebra*, Springer-Verlag, New York, 2005.
[12] S. Murai, N. Terai, h-vectors of simplicial complexes with Serre's conditions, *Math. Res. Lett.* **16** (2009), no. 6, 1015–1028.
[13] R. P. Stanley, *Combinatorics and Commutative Algebra*, Second Edition, Birkhäuser Boston, Inc., Boston, MA, 1996.
[14] M. L. Wachs, Whitney homology of semipure shellable posets, *J. Algebraic Combin.* **9** (1999), no. 2, 173–207.

M. R. Pournaki, Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
E-mail address: pournaki@ipm.ir
URL: http://math.ipm.ac.ir/pournaki/

S. A. Seyed Fakhari, Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran.
E-mail address: fakhari@ipm.ir
URL: http://math.ipm.ac.ir/fakhari/

S. Yassemi, School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
E-mail address: yassemi@ipm.ir
URL: http://math.ipm.ac.ir/yassemi/