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A Survey on the Estimation of Commutativity in Finite Groups

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Abstract. Let G be a finite group and let $\mathcal{C} = \{(x, y) \in G \times G \mid xy = yx\}$. Then $\Pr(G) = |\mathcal{C}|/|G|^2$ is the probability that two elements of G, chosen randomly with replacement, commute. This probability is a well known quantity, called commutativity degree of G, and indeed gives us an estimation of commutativity in G. In the last four decades this subject has enjoyed a flourishing development. In this article, we give a brief survey on the development of this subject and then we collect several of our results concerning $\Pr(G)$ as well as its various generalizations.

Keywords: Finite group; Group character; Commutativity degree.

1. Introduction

The study of group theoretical problems related to discrete probability has long been a topic of interest. Indeed, the story goes back to a sequence of papers all of them entitled "On some problems of a statistical group-theory" written by P.

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Erdős and P. Turán during the years 1965 to 1972 (see [7, 8, 9, 10, 11, 12, 13]). In Theorem IV of [10] Erdős and Turán have shown that in an arbitrary finite group G, the number of elements of

$$\mathcal{C} = \{(x, y) \in G \times G \mid xy = yx\}$$

is equal to $|G|k(G)$, where $k(G)$ is the number of conjugacy classes of G. Later, in 1973, W.H. Gustafson [18] reproved this theorem adopting the same technique which was used by Erdős and Turán [10]. Here, for convenience of the readers, we reproduce the proof as follows:

Note that

$$\mathcal{C} = \bigsqcup_{x \in G} \Big(\{x\} \times C_G(x) \Big),$$

where the symbol $\bigsqcup$ denotes the disjoint union of sets and $C_G(x)$ the centralizer of x in G. Therefore, we have

$$|\mathcal{C}| = \sum_{x \in G} \big| \{x\} \times C_G(x) \big| = \sum_{x \in G} \big| C_G(x) \big|.$$

Recall that if x and y are conjugate elements of G, then $C_G(x)$ and $C_G(y)$ are conjugate subgroups of G. Also the number of elements in the conjugacy class of x is equal to the index of $C_G(x)$ in G, i.e., $|G : C_G(x)|$. Therefore, if $x_1, \ldots, x_{k(G)}$ are representatives of the distinct conjugacy classes in G, then we have

$$|\mathcal{C}| = \sum_{i=1}^{k(G)} \big| G : C_G(x_i) \big| \big| C_G(x_i) \big| = \sum_{i=1}^{k(G)} \big| G \big| = |G|k(G),$$

as shown by Erdős, Turán, and Gustafson.

Gustafson in [18] has also asked the following question which not only formed the title of his paper but also became the leading problem for the subsequent development of the subject:

What is the probability that two group elements commute?

Using the above-mentioned argument, Gustafson himself has provided an elementary but significant answer to this question. More precisely, if $\Pr(G)$ denotes the probability that two elements of G, chosen randomly with replacement, commute, then

$$\Pr(G) = \frac{|\mathcal{C}|}{|G|^2} = \frac{|G|k(G)}{|G|^2} = \frac{k(G)}{|G|}.$$

The quantity $\Pr(G)$ has been termed by some authors as the *commutativity degree* of G. Indeed, it gives us an estimation of commutativity in finite groups. Clearly, a finite group G is abelian if and only if $\Pr(G) = 1$. Now, assuming G to be nonabelian, one can obtain (see [18]) an upper bound for $\Pr(G)$ as follows:

Consider the class equation of G given by

$$|G| = |Z(G)| + \sum_{i=1}^{t} |G : C_G(x_i)|,$$

where $Z(G)$ denotes the center of G and $x_1, \ldots, x_t$ are representatives of the distinct nontrivial conjugacy classes in G. For every i with $1 \leq i \leq t$, we have $|G : C_G(x_i)| \geq 2$ and so $t \leq (|G| - |Z(G)|)/2$. This implies that $k(G) = |Z(G)| + t \leq (|G| + |Z(G)|)/2$. Since G is nonabelian, it is easy to see that $G/Z(G)$ is not cyclic and so $|G/Z(G)| \geq 4$. Therefore, it follows that

$$\Pr(G) = \frac{k(G)}{|G|} \leq \frac{|G| + |Z(G)|}{2|G|} = \frac{1}{2} + \frac{|Z(G)|}{2|G|} \leq \frac{1}{2} + \frac{1}{8} = \frac{5}{8}.$$

Thus, in summary, given a finite group G, the probability $\Pr(G)$ that two of its elements, chosen randomly with replacement, commute equals $k(G)/|G|$, with $\Pr(G) = 1$ if and only if G is abelian and $\Pr(G) \leq 5/8$ if G is nonabelian. Further, as the group of quaternions of order eight shows, the upper bound $5/8$ is best possible. In [18], Gustafson has also shown, with a suitable interpretation, that this inequality remains valid for compact topological nonabelian groups. In the last four decades this subject has enjoyed a flourishing development. In the sequel we give a brief survey on the development of the subject.

In 1975, G. Sherman [33] generalized the above probability. Let G be a finite group acting on a finite set Ω. Put $\Pr_G(\Omega) = |\mathrm{Fix}(G, \Omega)|/|G||\Omega|$, where $\mathrm{Fix}(G, \Omega)$ is the set of pairs (g, ω) of $G \times \Omega$ such that $g.\omega = \omega$. Note that $\Pr_G(\Omega)$ is the probability that an element of G leaves an element of Ω fixed. Sherman [33] also considered the case for which G is abelian and A is the group of its automorphisms. $\Pr_A(G)$ is the probability that an automorphism leaves an element fixed. He proved that for groups of order p^n, p a prime, $\Pr_A(G) \leq 2(3/p^2)^{n/2}$. He further showed that if G_n is a sequence of abelian groups with $|G_n| \to \infty$, then $\Pr_A(G_n) \to 0$. Considering the action of G on itself by conjugation one obtains that $\Pr_G(G)$ is equal to $\Pr(G)$, i.e., the probability that two elements of G, chosen randomly with replacement, commute. In this sense $\Pr_G(\Omega)$ is a generalization of $\Pr(G)$.

In 1979, D.J. Rusin [32] continued the study of $\Pr(G)$. He considered, for a finite group G, two situations, namely, $G' \subseteq Z(G)$ and $G' \cap Z(G) = \{1\}$, where G' is the commutator subgroup of G. First, he obtained an explicit computation of $\Pr(G)$ for groups G with $G' \subseteq Z(G)$. In this case, G being the direct product of its Sylow subgroups, the product formula $\Pr(H \times K) = \Pr(H)\Pr(K)$ sufficed him to obtain Theorem 1 of [32]: *If G is a p-group with $G' \subseteq Z(G)$, then*

$$\Pr(G) = \frac{1}{|G'|}\left(1 + \sum_{G'/K \text{ cyclic}} \frac{(p-1)|G' : K|}{p^{n(K)+1}}\right),$$

with $n(K)$ defined by $|\{x \in G \mid gxg^{-1}x^{-1} \in K \ \forall g \in G\}| = |G|/p^{n(K)}$. In the second case, i.e., when $G' \cap Z(G) = \{1\}$, he showed that $\Pr(G) = \Pr(K)$ holds true for some groups K such that $K' \cong G'$ and $Z(K) = \{1\}$. On the other hand, for every group G, there exist at most a finite number of groups K with $K' \cong G'$ and $Z(K) = \{1\}$ (see [32, Proposition 4]). Under an additional condition that $|G'|$ is a prime, he obtained an explicit computation of $\Pr(G)$ in this second case as well. In Section IV, Rusin has classified the groups G for which $\Pr(G)$ is greater than $11/32$.

In 1995, P. Lescot [22] generalized $\Pr(G)$ in the other direction. Let G be a finite group and let $n \geq 0$ be an integer. Define

$$d_n(G) = \frac{|\{(x_1, \ldots, x_{n+1}) \in G^{n+1} \mid x_i x_j = x_j x_i,\ 1 \leq i, j \leq n+1\}|}{|G|^{n+1}}.$$

Note that $d_1(G)$ is equal to $\Pr(G)$, i.e., the probability that two elements of G, chosen randomly with replacement, commute. Therefore, this later quantity is a generalization of $\Pr(G)$. He showed that $d_n(G) = d_n(H)$ provided G and H are isoclinic in the sense of P. Hall (see [19, page 133]). He also showed that if G is a finite nonabelian group, then $d_n(G) \leq (3 \times 2^n - 1)/2^{2n+1}$. Moreover, the equality holds if and only if G is isoclinic to the group of quaternions of order eight. Also in 2001, Lescot [23] characterized all finite groups G with commutativity degree at least $1/2$. This is done by first determining all finite groups G whose central factor is a nonabelian group of order pq and then using a result of S.R. Blackburn [2] determining all finite 2-groups with derived subgroup of order two.

In 2008, M.R. Pournaki and R. Sobhani [31] gave a new generalization of $\Pr(G)$. For a finite group G and $g \in G$ define

$$\Pr_g(G) = \frac{|\{(x, y) \in G \times G \mid [x, y] = g\}|}{|G|^2},$$

where $[x, y]$ denotes the commutator of x and y. Note that $\Pr_1(G)$ is equal to $\Pr(G)$, i.e., the probability that two elements of G, chosen randomly with replacement, commute. Therefore, this later quantity is a generalization of $\Pr(G)$. They found some results concerning $\Pr_g(G)$ and after that A.K. Das and R.K. Nath in a sequence of different papers gave new results and other new generalizations regarding $\Pr(G)$ and $\Pr_g(G)$ (see [3, 4, 5, 6, 28, 29]).

In the rest of this article we collect several of our results concerning $\Pr(G)$ as well as its various generalizations.

2. Commutativity of Two Elements

In this section we deal with probability that two elements of a finite group G, chosen randomly with replacement, commute. As we mentioned in Section 1, this probability is $\Pr(G) = |\mathcal{C}|/|G|^2$, where $\mathcal{C} = \{(x, y) \in G \times G \mid xy = yx\}$. Here we collect several results dealing with this probability.

In 1979, Rusin [32] computed, for a finite group G, the values of $\Pr(G)$ when $G' \subseteq Z(G)$, and also when $G' \cap Z(G)$ is trivial. In this direction we have the following result.

Theorem 2.1. [5, Theorem 3.5] *Let G be a finite group and let p be a prime number such that* $\gcd(p-1, |G|) = 1$. *If $|G'| = p^2$ and $|G' \cap Z(G)| = p$, then the following statements hold:*

$$(1)\ \Pr(G) = \begin{cases} \frac{2p^2-1}{p^4} & \textit{if } C_G(G') \textit{ is abelian}, \\ \frac{1}{p^4}\left(\frac{p-1}{p^{2s-1}} + p^2 + p - 1\right) & \textit{otherwise}, \end{cases}$$

$$(2)\ \left|\frac{G}{Z(G)}\right| = \begin{cases} p^3 & \textit{if } C_G(G') \textit{ is abelian}, \\ p^{2s+2} \textit{ or } p^{2s+3} & \textit{otherwise}, \end{cases}$$

where $p^{2s} = |C_G(G') : Z(C_G(G'))|$. *Moreover, we have*

$$\left|\frac{G}{G' \cap Z(G)} : Z\Big(\frac{G}{G' \cap Z(G)}\Big)\right| = \left|\frac{G}{Z(G)} : Z\Big(\frac{G}{Z(G)}\Big)\right| = p^2.$$

It may be noted here that, on a number of occasions, the structure of $G/Z(G)$ is determined by its size. For example, using GAP [35] and the notion of semidirect product '$\rtimes$', one can see that if G is a finite group with $G' \nsubseteq Z(G)$, then $G/Z(G)$ is isomorphic to $C_7 \rtimes C_3$, $(C_3 \times C_3) \rtimes C_3$, $C_{13} \rtimes C_3$, $C_{19} \rtimes C_3$, $C_3 \times (C_7 \rtimes C_3)$, or $(C_5 \times C_5) \rtimes C_3$ according as $|G/Z(G)|$ is equal to 21, 27, 39, 57, 63, or 75. Here, C_n denotes the cyclic group of order n.

Rusin also determined all numbers lying in the interval $(11/32, 1]$ that can be realized as the commutativity degree of some finite groups, and also classified all finite groups whose commutativity degrees lie in the interval $(11/32, 1]$. In 2006, F. Barry, D. MacHale, and Á. Ní Shé [1] have shown that if G is a finite group with $|G|$ odd and $\Pr(G) > 11/75$, then G is supersolvable. They also proved that if $\Pr(G) > 1/3$, then G is supersolvable. It may be mentioned here that a group G is said to be *supersolvable* if there is a series of the form

$$\{1\} = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_r = G,$$

where $A_i \trianglelefteq G$ and A_{i+1}/A_i is cyclic for every i with $0 \leq i \leq r-1$ (see [16, 24, 34]). The previous theorem enables us to obtain the following characterization.

Theorem 2.2. [5, Theorem 4.3] *Let G be a finite group. If $|G|$ is odd and* $\Pr(G) \geq 11/75$, *then the possible values of* $\Pr(G)$ *and the corresponding structures of* G', $G' \cap Z(G)$, *and* $G/Z(G)$ *are given by the following* Table 1.

The following result gives a universal lower bound for $\Pr(G)$.

Theorem 2.3. [28, Theorem 1] *Let G be a finite group. Then we have*

$$\Pr(G) \geq \frac{1}{|G'|}\left(1 + \frac{|G'| - 1}{|G : Z(G)|}\right).$$

In particular, $\Pr(G) > 1/|G'|$ *provided G is nonabelian.*

There are several equivalent necessary as well as sufficient conditions for equality to hold in the above theorem. These are listed as follows.

$\Pr(G)$	G'	$G' \cap Z(G)$	$G/Z(G)$
1	$\{1\}$	$\{1\}$	$\{1\}$
$\frac{1}{3}(1+\frac{2}{3^{2s}})$	C_3	C_3	$(C_3 \times C_3)^s,\ s \geq 1$
$\frac{1}{5}(1+\frac{4}{5^{2s}})$	C_5	C_5	$(C_5 \times C_5)^s,\ s \geq 1$
$\frac{5}{21}$	C_7	$\{1\}$	$C_7 \rtimes C_3$
$\frac{55}{343}$	C_7	C_7	$C_7 \times C_7$
$\frac{17}{81}$	C_9 or $C_3 \times C_3$ or $C_3 \times C_3$	C_3 or $C_3 \times C_3$	$(C_3 \times C_3) \rtimes C_3$ or C_3^3
$\frac{121}{729}$	$C_3 \times C_3$	$C_3 \times C_3$	C_3^4
$\frac{7}{39}$	C_{13}	$\{1\}$	$C_{13} \rtimes C_3$
$\frac{3}{19}$	C_{19}	$\{1\}$	$C_{19} \rtimes C_3$
$\frac{29}{189}$	C_{21}	C_3	$C_3 \times (C_7 \rtimes C_3)$
$\frac{11}{75}$	$C_5 \times C_5$	$\{1\}$	$(C_5 \times C_5) \rtimes C_3$

Table 1: The possible values of $\Pr(G)$ and the corresponding structures of G', $G' \cap Z(G)$, and $G/Z(G)$ provided $|G|$ is odd and $\Pr(G) \geq 11/75$.

Theorem 2.4. [28, Theorem 2] *Let G be a finite nonabelian group. Then the following statements are equivalent:*

(1) *The equality* $\Pr(G) = \frac{1}{|G'|}\left(1 + \frac{|G'|-1}{|G:Z(G)|}\right)$ *holds.*

(2) *We have* $\mathrm{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$*, which means that G is of central type with $|\mathrm{cd}(G)| = 2$. Here, $\mathrm{cd}(G)$ denotes the set of irreducible complex character degrees of G.*

(3) *For every $x \in G \setminus Z(G)$, we have $|\mathcal{C}\ell_G(x)| = |G'|$. Here, $\mathcal{C}\ell_G(x)$ denotes the conjugacy class of x.*

(4) *For every $x \in G \setminus Z(G)$, we have $\mathcal{C}\ell_G(x) = G'x$. In particular, G is a nilpotent group of class 2.*

(5) *For every $x \in G \setminus Z(G)$, we have $C_G(x) \trianglelefteq G$ and $G' \cong G/C_G(x)$. In particular, G is a CN-group, i.e., a group in which the centralizer of every element is normal.*

(6) *For every $x \in G \setminus Z(G)$, we have $G' = \{[y,x] \mid y \in G\}$. In particular, every element of G' is a commutator.*

Theorem 2.3 and Theorem 2.4 not only allow us to obtain some characterizations for finite nilpotent groups of class 2 whose commutator subgroups have prime order, but also enable us to re-establish certain facts (essentially due to K.S. Joseph [21]) concerning the smallest prime divisors of the orders of finite groups.

Theorem 2.5. [28, Proposition 1] *Let G be a finite group and let p be the smallest prime divisor of $|G|$. Then the following statements hold:*

(1) *If $p \neq 2$, then $\Pr(G) \neq 1/p$.*

(2) *In the case of* G *is nonabelian,* $\Pr(G) > 1/p$ *if and only if* $|G'| = p$ *and* $G' \subseteq Z(G)$.

Some of the consequences of the above results are as given below.

Theorem 2.6. [28, Corollary 1] *Let* G *be a finite group with* $\Pr(G) = 1/3$*. Then* $|G|$ *is even.*

Theorem 2.7. [28, Corollary 2] *Let* G *be a finite group and let* $p \neq 2$ *be the smallest prime divisor of* $|G|$*. If* G *is nonabelian with* $G' \cap Z(G) = \{1\}$*, then* $\Pr(G) < 1/p$.

For a group K and an element k of that group consider $\overline{K} = K/Z(K)$ and $\overline{k} = kZ(K)$, and define the map $a_K : \overline{K} \times \overline{K} \longrightarrow K'$ by $a_K(\overline{x}, \overline{y}) = [x, y]$ which is obviously well defined. We now introduce the notion of isoclinism which is originally due to P. Hall (see [19, page 133]). Let G and H be two groups. A pair (φ, ψ) is called an *isoclinism* from G to H if φ is an isomorphism from $\overline{G}$ to $\overline{H}$ and ψ is an isomorphism from G' to H' for which the following diagram commutes.

$$\begin{array}{ccc} \overline{G} \times \overline{G} & \stackrel{\varphi\times\varphi}{\longrightarrow} & \overline{H} \times \overline{H} \\ \Big\downarrow a_G & & \Big\downarrow a_H \\ G' & \stackrel{\psi}{\longrightarrow} & H' \end{array}$$

If there is an isoclinism from G to H, we shall say that G and H are *isoclinic*. Clearly, isoclinism is an equivalence relation between groups. It is well known that if G and H are isomorphic groups, then they are isoclinic.

Theorem 2.8. [28, Proposition 2] *Let* G *be a finite group and let* p *be a prime. Then the following statements are equivalent:*

(1) *We have* $|G'| = p$ *and* $G' \subseteq Z(G)$.
(2) *The group* G *is of central type with* $|\mathrm{cd}(G)| = 2$ *and* $|G'| = p$.
(3) *The group* G *is a direct product of a* p*-group* P *and an abelian group* A *such that* $|P'| = p$ *and* $\gcd(p, |A|) = 1$.
(4) *The group* G *is isoclinic to an extra-special* p*-group, and consequently, the equality* $|G : Z(G)| = p^{2k}$ *holds true for some positive integer* k.

In particular, if G *is nonabelian and* p *is the smallest prime divisor of* $|G|$*, then the above statements are also equivalent to the condition* $\Pr(G) > 1/p$.

If $|G'| = p$, p prime, and $G' \subseteq Z(G)$, then we call the integer $2k + 1$, where k is as in Part 4 of Theorem 2.8, as the *isoclinic exponent* of G and denote it by $\mathrm{iso.exp}(G)$. Thus, we have, in particular, the following theorem:

Theorem 2.9. [31, Proposition 3.4] *Let G be a finite group such that $G' \subseteq Z(G)$ and let $|G'| = p$ be a prime. If* $\text{iso.exp}(G) = n$*, then*

$$\Pr(G) = \frac{1}{p}\left(1 + \frac{p-1}{p^{n-1}}\right).$$

Now consider a finite group G such that $|G'| = p$ is a prime and $G' \cap Z(G) = \{1\}$. We have $(G/Z(G))' \cong G'$. On the other hand, if we consider $Z(G/Z(G)) = H/Z(G)$ for some H, $Z(G) \leq H \leq G$, then $[G,\ H] \leq G' \cap Z(G) = \{1\}$ implies that $H = Z(G)$ and so we have $Z(G/Z(G)) = 1$. Therefore by [32, Proposition 5], there is a positive integer n depending only on G for which $G/Z(G) = \langle a, b \mid a^p = b^n = 1,\ bab^{-1} = a^r\rangle$. We call n as the *invariant number* of G and denote it by $\text{inv}(G)$.

Theorem 2.10. [31, Proposition 4.4] *Let G be a finite group such that $G' \cap Z(G) = \{1\}$ and let $|G'| = p$ be a prime. If* $\text{inv}(G) = n$*, then*

$$\Pr(G) = \frac{n^2 + p - 1}{pn^2}.$$

In [15, Corollary 1.2], I.V. Erovenko and B. Sury have shown, in particular, that for every integer $k > 1$ there exists a family $\{G_n\}$ of finite groups such that $\Pr(G_n) \to 1/k^2$ as $n \to \infty$. In this connection, we have the following observations.

Theorem 2.11. [27, Proposition 2.5.1] *For any $k \in \mathbb{N}$ with $k > 1$, there exists a family $\{G_n\}$ of finite groups such that* $\Pr(G_n) \to 1/k$ *as $n \to \infty$.*

Theorem 2.12. [27, Proposition 2.5.2] *For any $n \in \mathbb{N}$, there exists a finite group G such that* $\Pr(G) = 1/n$.

The following result gives us a formula as well as an upper bound for $\Pr(G)$ when G has exactly two irreducible complex character degrees.

Theorem 2.13. [31, Theorem 2.2] *Let G be a finite group such that* $\text{cd}(G) = \{1, m\}$, $m > 1$. *Then we have*

$$\Pr(G) = \frac{1}{|G'|}\left(1 + \frac{|G'| - 1}{m^2}\right) \leq \frac{|G'| + 3}{4|G'|}.$$

3. Commutators of Two Elements Which Are Equal to a Given Element

In this section we deal with the probability that the commutator of two elements, chosen randomly with replacement, in a finite group is equal to a given element of that group. For a finite group G and a given element g of G, this probability is $\mathrm{Pr}_g(G) = |\mathcal{C}_g|/|G|^2$, where $\mathcal{C}_g = \{(x, y) \in G \times G \mid [x, y] = g\}$. Clearly, this notion is a generalization of the pervious one, mentioned in Section 2, as we have $\mathrm{Pr}_1(G) = \mathrm{Pr}(G)$.

In the following we collect several results dealing with this probability. Obviously for those g's lie in $G \setminus G'$, we have $\mathrm{Pr}_g(G) = 0$. Therefore in the sequel we deal only with g's which lie in G'. Note that there are examples of groups G of order 96 due to R.M. Guralnick [17] where $\mathrm{Pr}_g(G) = 0$ even when g belongs to G'.

The first result shows that $\mathrm{Pr}_g(G)$ is an invariant under isoclinism in the following sense.

Theorem 3.1. [31, Lemma 3.5] *Let G and H be two isoclinic finite groups and let (φ, ψ) be an isoclinism from G to H. If $g \in G'$, then we have*

$$\mathrm{Pr}_g(G) = \mathrm{Pr}_{\psi(g)}(H).$$

The following result gives us a character theoretical formula for $\mathrm{Pr}_g(G)$.

Theorem 3.2. [31, Theorem 2.1] *Let G be a finite group and $g \in G'$. Then*

$$\mathrm{Pr}_g(G) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g)}{\chi(1)},$$

where Irr(G) *denotes the set of irreducible complex characters of G.*

We also derive a formula as well as a lower bound for $\mathrm{Pr}_g(G)$ when G has exactly two irreducible complex character degrees.

Theorem 3.3. [31, Theorem 2.2] *Let G be a finite group with* $\mathrm{cd}(G) = \{1, m\}$, *$m > 1$. If $g \in G'$, $g \neq 1$, then we have*

$$\mathrm{Pr}_g(G) = \frac{1}{|G'|}\left(1 - \frac{1}{m^2}\right) \geq \frac{3}{4|G'|}.$$

In view of [20, Problem 2.13], we obtain the following theorem.

Theorem 3.4. [31, Propositions 3.1 and 3.4] *Let G be a finite group such that*

$G' \subseteq Z(G)$ and let $|G'| = p$ be a prime. If $g \in G'$, $g \neq 1$, then we have

$$\Pr_g(G) = \frac{1}{p}\left(1 - \frac{1}{|G : Z(G)|}\right) \geq \frac{3}{4p}.$$

In particular, if iso.exp$(G) = n$*, then we obtain that*

$$\Pr_g(G) = \frac{1}{p}\left(1 - \frac{1}{p^{n-1}}\right).$$

A universal upper bound for $\Pr_g(G)$, $g \in G'$, $g \neq 1$, is given as follows.

Theorem 3.5. [31, Corollary 2.3] *Let G be a finite group with* $|\text{cd}(G)| = 2$*. Let $g \in G'$, $g \neq 1$. Then we have*

$$\Pr_g(G) \leq \frac{1}{|G'|}\left(1 - \frac{1}{|G : Z(G)|}\right).$$

Moreover, the equality holds if and only if G is of central type.

The following result gives us a formula for $\Pr_g(G)$, $g \in G'$, $g \neq 1$, when $|G'|$ is prime and $G' \cap Z(G)$ is trivial.

Theorem 3.6. [31, Proposition 4.4] *Let G be a finite group such that $G' \cap Z(G) = \{1\}$ and let $|G'| = p$ be a prime. If* inv$(G) = n$ *and $g \in G'$, $g \neq 1$, then we have*

$$\Pr_g(G) = \frac{n^2 - 1}{pn^2}.$$

We close this section by listing some more properties of $\Pr_g(G)$.

Theorem 3.7. [31, Proposition 5.1] *Let G be a finite group and let $g \in G'$. Then we have $\Pr_g(G) \leq \Pr(G)$. Moreover, the equality holds if and only if $g = 1$.*

Theorem 3.8. [31, Proposition 5.2] *Let G be a finite group and let $g \in G'$, $g \neq 1$. Then we have $\Pr_g(G) < 1/2$.*

Theorem 3.9. [31, Proposition 5.3] *For any $\epsilon \in \mathbb{R}$ with $\epsilon > 0$, there exists a finite group G and $g \in G$ such that $1/2 - \epsilon < \Pr_g(G) < 1/2$.*

4. Admissible Words Which Are Equal to a Given Element

In this section we deal with a probability which is a generalization of the previous one mentioned in Section 3. For a given positive integer n, consider the free group of words on n generators $x_1, x_2, \dots, x_n$. A word $\omega(x_1, x_2, \dots, x_n)$ is called *admissible* (see [3]) if each letter in it has precisely two nonzero indices, namely, $+1$ and -1. Given a nontrivial admissible word $\omega(x_1, x_2, \dots, x_n)$, $n \geq 2$, and an element $g \in G$, consider the ratio

$$\mathrm{Pr}_g^\omega(G) = \frac{|\{(g_1, g_2, \dots, g_n) \in G^n \mid \omega(g_1, g_2, \dots, g_n) = g\}|}{|G^n|}.$$

Note that, for $\omega(x_1, x_2) = x_1 x_2 x_1^{-1} x_2^{-1}$, we have $\mathrm{Pr}_g^\omega(G) = \mathrm{Pr}_g(G)$, and so this later probability is a generalization of $\mathrm{Pr}_g(G)$.

In the following we collect several results dealing with this probability.

Theorem 4.1. [6, Proposition 2.2] *Let G be a finite group and let $\omega(x_1, x_2, \dots, x_n)$ be a nontrivial admissible word. Then the following statements hold:*

(1) $\mathrm{Pr}_1^\omega(G) = 1$ *if and only if G is abelian.*
(2) $\mathrm{Pr}_g^\omega(G) = \mathrm{Pr}_h^\omega(G)$ *if $g, h \in G'$ are conjugate in G.*
(3) $\mathrm{Pr}_g^\omega(G) = \mathrm{Pr}_h^\omega(G)$ *if $g, h \in G'$ generate the same cyclic subgroups of G. Consequently, we obtain that*

$$\mathrm{Pr}_g^\omega(G) \;=\; \frac{1 - \mathrm{Pr}_1^\omega(G)}{p - 1}$$

provided $|G'| = p$ is a prime and $g \in G'$, $g \neq 1$.

The quantity $\mathrm{Pr}_g^\omega(G)$ respects the cartesian product in the following sense.

Theorem 4.2. [6, Proposition 2.3] *Let H and K be two finite groups, let $(h, k) \in H' \times K'$ and $\omega(x_1, x_2, \dots, x_n)$ be a nontrivial admissible word. Then we have $\mathrm{Pr}_{(h,k)}^\omega(H \times K) = \mathrm{Pr}_h^\omega(H)\mathrm{Pr}_k^\omega(K)$.*

The following result shows that $\mathrm{Pr}_g^\omega(G)$ is an invariant under isoclinism of finite groups.

Theorem 4.3. [6, Proposition 2.4] *Let G and H be two finite groups and (ϕ, ψ) be an isoclinism from G to H. If $g \in G'$ and $\omega(x_1, x_2, \dots, x_n)$ is a nontrivial admissible word, then we have $\mathrm{Pr}_g^\omega(G) = \mathrm{Pr}_{\psi(g)}^\omega(H)$.*

Theorem 4.4. [6, Proposition 3.1] *Let G be a finite group and let $\omega(x_1, x_2, \dots, x_n)$ be a nontrivial admissible word. If $g \in G'$, then we have*

$$\frac{\mathrm{Pr}_g(G)}{|G : Z(G)|^{n-2}} \leq \mathrm{Pr}_g^\omega(G).$$

If G is a finite nonabelian simple group and $\omega(x_1, x_2, \dots, x_n)$ is a nontrivial admissible word, then using Ore conjecture (see [30]), which has been established recently in [25, Theorem 1], it follows from the above result that every element of G is of the form $\omega(g_1, g_2, \ldots, g_n)$ for some $g_1, g_2, \ldots, g_n \in G$.

As a generalization of Theorem 3.7, we have the following result.

Theorem 4.5. [6, Proposition 3.6] *Let G be a finite group and let $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. If $g \in G'$, then the following statements hold:*
(1) $\Pr_g^{\omega}(G) \le \Pr_1^{\omega}(G) \le \Pr(G)$.
(2) $\Pr_g^{\omega}(G) = \Pr_1^{\omega}(G)$ *if and only if* $g = 1$.
Consequently, $\Pr_g^{\omega}(G) = 1$ *if and only if* $g = 1$ *and* G *is abelian.*

The following result gives us a universal lower bound for $\Pr_1^{\omega}(G)$.

Theorem 4.6. [6, Proposition 3.7] *Let G be a finite group and let $\omega(x_1, x_2, \ldots, x_n)$ be a nontrivial admissible word. Then we have*

$$\Pr_1^{\omega}(G) \ge \frac{1}{|G'|}\left(1 + \frac{|G'| - 1}{|G : Z(G)|^{n/2}}\right).$$

In particular, $\Pr_1^{\omega}(G) > \frac{1}{|G'|}$ *provided G is nonabelian.*

The following result generalizes Theorem 3.8.

Theorem 4.7. [6, Proposition 3.8] *Let G be a finite nonabelian group, let $g \in G'$ and p be the smallest prime divisor of $|G|$. If $\omega(x_1, x_2, \ldots, x_n)$ is a nontrivial admissible word and $g \ne 1$, then $\Pr_g^{\omega}(G) < 1/p$. In particular, we have $\Pr_g^{\omega}(G) < 1/2$.*

Let us now take the nontrivial admissible word $\omega(x_1, x_2, \ldots, x_n)$ to be $x_1 x_2 \ldots x_n x_1^{-1} x_2^{-1} \ldots x_n^{-1}$, and write $\Pr_g^n(G)$ in place of $\Pr_g^{\omega}(G)$. It has been observed that $\Pr_g^n(G) = \Pr_g^{n+1}(G)$. Hence, without any loss we may assume that n is even.

Theorem 4.8. [29, Equations 8, 11, and 12] *Let G be a finite nonabelian group, let $g \in G'$ and $2 \le d \le \min\{\chi(1) \mid \chi \in \mathrm{Irr}(G),\ \chi(1) \ne 1\}$. Then the following statements hold:*
(1) $\left|\Pr_g^n(G) - \frac{1}{|G'|}\right| \le \frac{1}{d^{n-2}}\left(\Pr(G) - \frac{1}{|G'|}\right)$.
(2) $\left|\Pr_g^n(G) - \frac{1}{|G'|}\right| \le \frac{1}{d^{n}}\left(1 - \frac{1}{|G'|}\right)$. *In particular, we have* $\Pr_g^n(G) \le \frac{2^n+1}{2^{n+1}}$.

Theorem 4.9. [29, Equation 14] *Let G be a finite nonabelian simple group and let $g \in G$. Then we have*

$$\left|\Pr_g^n(G) - \frac{1}{|G|}\right| \le \frac{1}{3^{n-2}}\left(\frac{1}{12} - \frac{1}{|G|}\right).$$

In particular, we conclude that

$$\Pr_g^n(G) \leq \frac{3^{n-2}+4}{3^{n-1}\times 20}.$$

The following result generalizes Theorem 3.3.

Theorem 4.10. [29, Proposition 5.1] *Let G be a finite nonabelian group and let $g \in G'$, $g \neq 1$. If $\mathrm{cd}(G) = \{1, d\}$, $d > 1$, then we have*

$$\Pr_g^n(G) = \frac{1}{|G'|}\left(1 - \frac{1}{d^n}\right).$$

The next two results give us some necessary and sufficient conditions for equality to hold in Theorem 4.8.

Theorem 4.11. [29, Propositions 4.2 and 5.2] *Let G be a finite nonabelian group, let $g \in G'$ and $2 \leq d \leq \min\{\chi(1) \mid \chi \in \mathrm{Irr}(G),\ \chi(1) \neq 1\}$. Then the following statements hold:*

(1) $\Pr_g^n(G) = \frac{1}{d^{n-2}}\left(\Pr(G) + \frac{d^{n-2}-1}{|G'|}\right)$ *if and only if* $g = 1$ *and* $\mathrm{cd}(G) = \{1, d\}$.

(2) $\Pr_g^n(G) = \frac{1}{d^{n-2}}\left(-\Pr(G) + \frac{d^{n-2}+1}{|G'|}\right)$ *if and only if* $g \neq 1$, $\mathrm{cd}(G) = \{1, d\}$, *and* $|G'| = 2$.

Theorem 4.12. [29, Proposition 4.4 and Corollary 5.3] *Let G be a finite nonabelian group, let $g \in G'$ and $2 \leq d \leq \min\{\chi(1) \mid \chi \in \mathrm{Irr}(G),\ \chi(1) \neq 1\}$. Then the following statements hold:*

(1) $\Pr_g^n(G) = \frac{1}{d^n}\left(1 + \frac{d^n-1}{|G'|}\right)$ *if and only if* $g = 1$ *and* $\mathrm{cd}(G) = \{1, d\}$.

(2) $\Pr_g^n(G) = \frac{1}{d^n}\left(-1 + \frac{d^n+1}{|G'|}\right)$ *if and only if* $g \neq 1$, $\mathrm{cd}(G) = \{1, d\}$, *and* $|G'| = 2$.

The following result provides a characterization of finite groups up to isoclinism.

Theorem 4.13. [29, Proposition 4.6] *Let G be a finite nonabelian group, let $g \in G'$ and p be the smallest prime divisor of $|G|$. Then we have*

$$\Pr_g^n(G) = \frac{p^n+p-1}{p^{n+1}},$$

if and only if $g = 1$ *and* G *is isoclinic to* $\langle x, y \mid x^{p^2} = 1 = y^p,\ y^{-1}xy = x^{p+1}\rangle$. *In particular, putting* $p = 2$, *we conclude that*

$$\mathrm{Pr}_g^n(G) = \frac{2^n + 1}{2^{n+1}},$$

if and only if $g = 1$ *and* G *is isoclinic to* D_8, *the dihedral group, and hence, to* Q_8, *the group of quaternions.*

We now list a few results which are basically generalizations of some of the results obtained in [31].

Theorem 4.14. [29, Proposition 6.1] *Let* G *be a finite nonabelian group with* $|\mathrm{cd}(G)| = 2$ *and let* $g \in G'$. *Then we have*

$$\mathrm{Pr}_1^n(G) \geq \frac{1}{|G'|}\left(1 + \frac{|G'| - 1}{|G : Z(G)|^{n/2}}\right) \text{ and}$$
$$\mathrm{Pr}_g^n(G) \leq \frac{1}{|G'|}\left(1 - \frac{1}{|G : Z(G)|^{n/2}}\right) \text{ provided } g \neq 1.$$

Moreover, in each case, the equality holds if and only if G *is of central type.*

Theorem 4.15. [29, Corollary 6.2] *Let* G *be a finite nonabelian group and* $g \in G'$. *Let* G *be of central type with* $|\mathrm{cd}(G)| = 2$. *Then we have*

$$\mathrm{Pr}_1^n(G) \leq \frac{1}{|G'|}\left(1 + \frac{|G'| - 1}{2^n}\right) \text{ and}$$
$$\mathrm{Pr}_g^n(G) \geq \frac{1}{|G'|}\left(1 - \frac{1}{2^n}\right) \text{ provided } g \neq 1.$$

Theorem 4.16. [29, Proposition 6.3] *Let* G *be a finite nonabelian group and let* $G' \subseteq Z(G)$ *and* $|G'| = p$ *be a prime. If* $g \in G'$, *then we have*

$$\mathrm{Pr}_g^n(G) = \begin{cases} \frac{1}{p}\left(1 + \frac{p-1}{p^{nk}}\right) & \text{if } g = 1, \\ \frac{1}{p}\left(1 - \frac{1}{p^{nk}}\right) & \text{if } g \neq 1, \end{cases}$$

where $k = \frac{1}{2}\log_p |G : Z(G)|$.

Theorem 4.17. [29, Proposition 6.5] *Let* G *be a finite nonabelian group and* $g \in G'$. *If* $G' \cap Z(G) = \{1\}$, $|G'| = p$, *where* p *is a prime, and* $\mathrm{inv}(G) = r$, *then we have*

$$\mathrm{Pr}_g^n(G) = \begin{cases} \frac{r^n + p - 1}{pr^n} & \text{if } g = 1, \\ \frac{r^n - 1}{pr^n} & \text{if } g \neq 1. \end{cases}$$

Theorem 4.18. [29, Proposition 6.7] *For any $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ and for any prime number p, there exists a finite group G such that the inequality*

$$\left|\mathrm{Pr}_g^n(G) - \frac{1}{p}\right| < \varepsilon$$

holds true for all $g \in G'$.

5. Commutators of Two Subgroups Which Are Equal to a Given Element

In 2007, A. Erfanian, R. Rezaei, and P. Lescot [14] studied the probability $\mathrm{Pr}(H, G)$ that an element of a given subgroup H of a finite group G commutes with an element of G (see also [26]). Note that $\mathrm{Pr}(G, G) = \mathrm{Pr}(G)$. This notion has been further generalized as follows. Let G be a finite group and $g \in G'$. Let H and K be two subgroups of G. Consider the ratio

$$\mathrm{Pr}_g(H, K) = \frac{|\{(x, y) \in H \times K \mid [x, y] = g\}|}{|H||K|}.$$

If $g = 1$, then for brevity we write $\mathrm{Pr}_1(H, K) = \mathrm{Pr}(H, K)$. Note that for $H = K = G$, we have $\mathrm{Pr}_g(H, K) = \mathrm{Pr}_g(G)$.

The following theorem says that $\mathrm{Pr}_g(H, K)$ is not very far from being symmetric with respect to H and K.

Theorem 5.1. [4, Proposition 2.1] *Let G be a finite group and let $g \in G'$. If H and K are two subgroups of G, then we have $\mathrm{Pr}_g(H, K) = \mathrm{Pr}_{g^{-1}}(K, H)$. However, if $g^2 = 1$, or, if $g \in H \cup K$ (for example, when H or K is normal in G), then we have $\mathrm{Pr}_g(H, K) = \mathrm{Pr}_g(K, H) = \mathrm{Pr}_{g^{-1}}(H, K)$.*

The quantity $\mathrm{Pr}_g(H, K)$ respects the cartesian product in the following sense.

Theorem 5.2. [4, Proposition 2.2] *Let G_1 and G_2 be two finite groups with subgroups $H_1, K_1 \subseteq G_1$ and $H_2, K_2 \subseteq G_2$. Let $g_1 \in G_1'$ and $g_2 \in G_2'$. Then we have $\mathrm{Pr}_{(g_1,g_2)}(H_1 \times H_2, K_1 \times K_2) = \mathrm{Pr}_{g_1}(H_1, K_1)\mathrm{Pr}_{g_2}(H_2, K_2)$.*

Now, we have the following computing formula which plays a key role in the study of $\mathrm{Pr}_g(H, K)$.

Theorem 5.3. [4, Theorem 2.3] *Let G be a finite group and let $g \in G'$. If H and K are two subgroups of G, then we have*

$$\mathrm{Pr}_g(H, K) = \frac{1}{|H||K|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathcal{C}\ell_K(x)}} |C_K(x)| = \frac{1}{|H|} \sum_{\substack{x \in H \\ g^{-1}x \in \mathcal{C}\ell_K(x)}} \frac{1}{|\mathcal{C}\ell_K(x)|},$$

where $C_K(x) = \{y \in K \mid xy = yx\}$ *and* $\mathcal{C}\ell_K(x) = \{yxy^{-1} \mid y \in K\}$*, the* $K-$*conjugacy class of* x*.*

The following result generalizes the well known formula $\Pr(G) = k(G)/|G|$.

Theorem 5.4. [4, Corollary 2.4] *Let* G *be a finite group and let* H *and* K *be two subgroups of* G*. If* $H \trianglelefteq G$*, then we have*

$$\Pr(H, K) = \frac{k_K(H)}{|H|},$$

where $k_K(H)$ *is the number of* $K-$*conjugacy classes that constitute* H*.*

Let G be a finite group. If $H \trianglelefteq G$ with $C_G(x) \subseteq H$ for all $x \in H \setminus \{1\}$, then using Sylow's theorems and the fact that nontrivial p-groups have nontrivial centers, we have $\gcd(|H|, |G : H|) = 1$. Therefore, by the Schur-Zassenhaus theorem, H has a complement in G. Such groups belong to a well known class of groups called the Frobenius groups; for example, the alternating group A_4, the dihedral groups of order $2n$ with n odd, the nonabelian groups of order pq, where p and q are primes with $q|(p-1)$, etc.

Theorem 5.5. [4, Proposition 2.5] *Let* G *be a finite group. If* H *is an abelian normal subgroup of* G *with a complement* K *in* G *and* $g \in G'$*, then we have* $\Pr_g(H, G) = \Pr_g(H, K)$*.*

As a consequence, we conclude the following theorem.

Theorem 5.6. [4, Corollary 2.6] *Let* G *be a finite group and let* $g \in G'$*. If* $H \trianglelefteq G$ *with* $C_G(x) = H$ *for all* $x \in H \setminus \{1\}$*, then we have* $\Pr_g(H, G) = \Pr_g(H, K)$*, where* K *is a complement of* H *in* G*. In particular,*

$$\Pr(H, G) = \frac{1}{|H|} + \frac{|H| - 1}{|G|}.$$

The following result gives us some conditional lower bounds for $\Pr_g(H, K)$.

Theorem 5.7. [4, Proposition 3.1] *Let* G *be a finite group and* $g \in G'$*. Let* H *and* K *be any two subgroups of* G*. If* $g \neq 1$*, then the following statements hold:*

(1) *If* $\Pr_g(H, K) \neq 0$ *then* $\Pr_g(H, K) \geq \frac{|C_H(K)||C_K(H)|}{|H||K|}$.
(2) *If* $\Pr_g(H, G) \neq 0$ *then* $\Pr_g(H, G) \geq \frac{2|H \cap Z(G)||C_G(H)|}{|H||G|}$.
(3) *If* $\Pr_g(G) \neq 0$ *then* $\Pr_g(G) \geq \frac{3}{|G:Z(G)|^2}$.

The next two results are generalizations of Theorem 3.7 and Theorem 3.8.

Theorem 5.8. [4, Proposition 3.2] *Let G be a finite group and let $g \in G'$. If H and K are any two subgroups of G, then* $\Pr_g(H,K) \leq \Pr(H,K)$. *Moreover, the equality holds if and only if $g = 1$.*

Theorem 5.9. [4, Proposition 3.3] *Let G be a finite group and $g \in G'$, $g \neq 1$. Let H and K be any two subgroups of G. If p is the smallest prime divisor of $|G|$, then we have*

$$\Pr_g(H,K) \leq \frac{|H| - |C_H(K)|}{p|H|} < \frac{1}{p}.$$

The quantity $\Pr_g(H,K)$ is monotonic in the following sense.

Theorem 5.10. [4, Proposition 3.4] *Let G be a finite group. Let H, K_1, and K_2 be three subgroups of G with $K_1 \subseteq K_2$. Then we have* $\Pr(H,K_1) \geq \Pr(H,K_2)$. *Moreover, the equality holds if and only if $\mathcal{C}\ell_{K_1}(x) = \mathcal{C}\ell_{K_2}(x)$ for all $x \in H$.*

Theorem 5.11. [4, Proposition 3.5] *Let G be a finite group. Let H, K_1, and K_2 be three subgroups of G with $K_1 \subseteq K_2$. Then we have*

$$\Pr(H,K_2) \geq \frac{1}{|K_2 : K_1|}\left(\Pr(H,K_1) + \frac{|K_2| - |K_1|}{|H||K_1|}\right).$$

Moreover, the equality holds if and only if $C_H(x) = \{1\}$ for all $x \in K_2 \setminus K_1$.

Theorem 5.12. [4, Proposition 3.6] *Let G be a finite group. Let $H_1 \subseteq H_2$ and $K_1 \subseteq K_2$ be subgroups of G and $g \in G'$. Then we have* $\Pr_g(H_1,K_1) \leq |H_2 : H_1||K_2 : K_1|\Pr_g(H_2,K_2)$. *Moreover, the equality holds if and only if $g^{-1}x \notin \mathcal{C}\ell_{K_2}(x)$ for all $x \in H_2 \setminus H_1, g^{-1}x \notin \mathcal{C}\ell_{K_2}(x) \setminus \mathcal{C}\ell_{K_1}(x)$ for all $x \in H_1$, and $C_{K_1}(x) = C_{K_2}(x)$ for all $x \in H_1$ with $g^{-1}x \in \mathcal{C}\ell_{K_1}(x)$. In particular, for $g = 1$, the condition for equality reduces to $H_1 = H_2$ and $K_1 = K_2$.*

The following result also generalizes Theorem 3.7 in some sense.

Theorem 5.13. [4, Corollary 3.7] *Let G be a finite group, let H be a subgroup of G and $g \in G'$. Then we have* $\Pr_g(H,G) \leq |G : H|\Pr(G)$. *Moreover, the equality holds if and only if $g = 1$ and $H = G$.*

We continue the survey by mentioning a few generalizations of some results obtained in [14].

Theorem 5.14. [4, Theorem 3.8] *Let G be a finite group and let p be the smallest* *prime divisor of* $|G|$*. If* H *and* K *are any two subgroups of* G*, then we have*

$$\Pr(H,K) \geq \frac{|C_H(K)|}{|H|} + \frac{p(|H| - |X_H| - |C_H(K)|) + |X_H|}{|H||K|} \quad \text{and}$$
$$\Pr(H,K) \leq \frac{(p-1)|C_H(K)| + |H|}{p|H|} - \frac{|X_H|(|K| - p)}{p|H||K|},$$

where $X_H = \{x \in H \mid C_K(x) = \{1\}\}$*. Moreover, in each of these bounds,* H *and* K *can be interchanged.*

Theorem 5.15. [4, Corollary 3.9] *Let* G *be a finite group and let* p *be the smallest prime divisor of* $|G|$*. If* H *and* K *are two subgroups of* G *such that* $[H,K] \neq \{1\}$*, then we have*

$$\Pr(H,K) \leq \frac{2p-1}{p^2}.$$

In particular, we conclude that $\Pr(H,K) \leq 3/4$*.*

Theorem 5.16. [4, Proposition 3.10] *Let* G *be a finite group and let* H *and* K *be any two subgroups of* G*. If* $\Pr(H,K) = (2p-1)/p^2$ *for some prime* p*, then* p *divides* $|G|$*. If* p *happens to be the smallest prime divisor of* $|G|$*, then we have*

$$\frac{H}{C_H(K)} \cong C_p \cong \frac{K}{C_K(H)}, \text{ and hence, } H \neq K.$$

In particular, we conclude that

$$\frac{H}{C_H(K)} \cong C_2 \cong \frac{K}{C_K(H)} \text{ provided } \Pr(H,K) = \frac{3}{4}.$$

We conclude the survey with the following result, which gives us a character theoretical formula.

Theorem 5.17. [4, Theorem 4.2 and Proposition 4.4] *Let* G *be a finite group. If* H *is a normal subgroup of* G *and* $g \in G'$*, then we have*

$$\Pr_g(H,G) = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{[\chi_H, \chi_H]}{\chi(1)} \chi(g).$$

Consequently, we conclude that

$$\left| \Pr_g(H,G) - \frac{1}{|G'|} \right| \leq |G:H| \left(\Pr(G) - \frac{1}{|G'|} \right).$$

The above result yields, in particular, that if G is a finite group with $|G'| \leq p^2$, where p is the smallest prime divisor of $|G|$, then every element of G' is a commutator.

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References

[1] F. Barry, D. MacHale, Á. Ní Shé, Some supersolvability conditions for finite groups, *Math. Proc. R. Ir. Acad.* **106A** (2) (2006) 163–177.

[2] S.R. Blackburn, Groups of prime power order with derived subgroup of prime order, *J. Algebra* **219** (2) (1999) 625–657.

[3] A.K. Das, R.K. Nath, On solutions of a class of equations in a finite group, *Comm. Algebra* **37** (11) (2009) 3904–3911.

[4] A.K. Das, R.K. Nath, On generalized relative commutativity degree of a finite group, *Int. Electron. J. Algebra* **7** (2010) 140–151.

[5] A.K. Das, R.K. Nath, A characterization of certain finite groups of odd order, *Math. Proc. R. Ir. Acad.* **111A** (2) (2011) 69–78.

[6] A.K. Das, R.K. Nath, A generalization of commutativity degree of finite groups, *Comm. Algebra* **40** (6) (2012) 1974–1981.

[7] P. Erdős, P. Turán, On some problems of a statistical group-theory I, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** (1965) 175–186.

[8] P. Erdős, P. Turán, On some problems of a statistical group-theory II, *Acta Math. Acad. Sci. Hungar.* **18** (1967) 151–163.

[9] P. Erdős, P. Turán, On some problems of a statistical group-theory III, *Acta Math. Acad. Sci. Hungar.* **18** (1967) 309–320.

[10] P. Erdős, P. Turán, On some problems of a statistical group-theory IV, *Acta Math. Acad. Sci. Hungar.* **19** (1968) 413–435.

[11] P. Erdős, P. Turán, On some problems of a statistical group-theory V, *Period. Math. Hungar.* **1** (1) (1971) 5–13.

[12] P. Erdős, P. Turán, On some problems of a statistical group-theory VI, *J. Indian Math. Soc.* **34** (3-4) (1970) 175–192.

[13] P. Erdős, P. Turán, On some problems of a statistical group-theory VII, *Period. Math. Hungar.* **2** (1972) 149–163.

[14] A. Erfanian, R. Rezaei, P. Lescot, On the relative commutativity degree of a subgroup of a finite group, *Comm. Algebra* **35** (12) (2007) 4183–4197.

[15] I.V. Erovenko, B. Sury, Commutativity degrees of wreath products of finite abelian groups, *Bull. Aust. Math. Soc.* **77** (1) (2008) 31–36.

[16] X. Guo, X. Sun, K.P. Shum, On the solvability of certain c-supplemented finite groups, *Southeast Asian Bull. Math.* **28** (6) (2004) 1029–1040.

[17] R.M. Guralnick, Commutators and commutator subgroups, *Adv. in Math.* **45** (3) (1982) 319–330.

[18] W.H. Gustafson, What is the probability that two group elements commute? *Amer. Math. Monthly* **80** (1973) 1031–1034.

[19] P. Hall, The classification of prime-power groups, *J. Reine Angew. Math.* **182** (1940) 130–141.

[20] I.M. Isaacs, *Character Theory of Finite Groups*, Dover Publications Inc., New York, 1994.

[21] K.S. Joseph, *Commutativity in Non-Abelian Groups*, Ph.D. thesis, University of California, Los Angeles, 1969.

[22] P. Lescot, Isoclinism classes and commutativity degrees of finite groups, *J. Algebra* **177** (3) (1995) 847–869.

[23] P. Lescot, Central extensions and commutativity degree, *Comm. Algebra* **29** (10) (2001) 4451–4460.

[24] S. Li, On s-completions of finite groups, *Southeast Asian Bull. Math.* **30** (2) (2006) 277–282.

[25] M.W. Liebeck, E.A. ÓBrien, A. Shalev, P.H. Tiep, The Ore conjecture, *J. Eur. Math. Soc.* **12** (4) (2010) 939–1008.

[26] X.L. Luo, X.Y. Guo, Finite groups in which every non-abelian subgroup is s-permutable, *Southeast Asian Bull. Math.* **33** (6) (2009) 1143–1147.

[27] R.K. Nath, *Commutativity Degree, Its Generalizations, and Classification of Finite Groups*, Ph.D. thesis, North-Eastern Hill University, India, 2010.

[28] R.K. Nath, A.K. Das, On a lower bound of commutativity degree, *Rend. Circ. Mat. Palermo (2)* **59** (1) (2010) 137–142.

[29] R.K. Nath, A.K. Das, On generalized commutativity degree of a finite group, *Rocky Mountain J. Math.* **41** (6) (2011) 1987–2000.

[30] O. Ore, Some remarks on commutators, *Proc. Amer. Math. Soc.* **2** (1951) 307–314.

[31] M.R. Pournaki, R. Sobhani, Probability that the commutator of two group elements is equal to a given element, *J. Pure Appl. Algebra* **212** (4) (2008) 727–734.

[32] D.J. Rusin, What is the probability that two elements of a finite group commute? *Pacific J. Math.* **82** (1) (1979) 237–247.

[33] G. Sherman, What is the probability an automorphism fixes a group element? *Amer. Math. Monthly* **82** (1975) 261–264.

[34] Z. Wu, Modules over restricted supersolvable Lie algebras, *Southeast Asian Bull. Math.* **27** (6) (2004) 1123–1128.

[35] The Gap Group, Gap–Groups, Algorithms, and Programming, http://www.gap-system.org.