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# Pure-injectivity of Tensor Products of Modules*

(Dedicated with gratitude to Edgar E. Enochs, our teacher and friend)

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**Abstract.** A classical question of Yoneda asks when the tensor product of two injective modules is injective. A complete answer to this question was given by Enochs and Jenda in 1991. In this paper the analogue question for pure-injective modules is studied.

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## 1 Introduction

Throughout the paper, all rings are commutative with nonzero identity and all modules are assumed to be unitary left modules. The theory of injective modules has

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a substantial role to play in all aspects of algebra. There are several generalizations of injective modules. Among these generalizations, pure-injective modules are a topic of interest.

Let $R$ be a ring and let $M, N$ be $R$-modules. An injective $R$-homomorphism $\varphi : M \to N$ is called *pure* whenever for every $R$-module $L$, the $R$-homomorphism $\varphi \otimes 1_L : M \otimes_R L \to N \otimes_R L$ is injective. An $R$-module $P$ is called *pure-injective* whenever for every pure $R$-homomorphism $\varphi : M \to N$, the induced $R$-homomorphism $\tilde{\varphi} : \mathrm{Hom}_R(N, P) \to \mathrm{Hom}_R(M, P)$ is surjective. In model theory, pure-injective modules are more useful than the injective modules. Also, there are some excellent applications of this notion in the theory of flat covers. Therefore, this notion has attracted more notice in recent years. For a survey on pure-injective modules, we refer the reader to [15] and [5], and as a general reference on the subject to [13].

A classical question of the Japanese mathematician Yoneda asks when the tensor product of two injective modules is injective. One of the first partial results related to this question was given by Hattori [7] in 1957. He showed that the tensor product of two injective $R$-modules is injective when $R$ is an integral domain. More generally, one can easily check that Hattori's result is also true when $R$ is the product of a finite number of integral domains. Later, in 1965, Ishikawa [8] showed that if $R$ is a Noetherian ring and the injective envelope $\mathrm{E}(R)$ of $R$ is a flat $R$-module, then the tensor product of two injective $R$-modules is injective. Finally, a complete answer to the question of Yoneda was given by Enochs and Jenda in 1991 by showing the converse of Ishikawa's result (see [3, Theorem 2.4]). In fact, they showed that if the tensor product of two injective $R$-modules is injective, then the injective envelope $\mathrm{E}(R)$ of $R$ is flat. For this part, the fact is used that $\mathrm{E}(R)$ is flat if and only if for every $\mathfrak{p} \in \mathrm{Ass}(R)$, the ring $R_{\mathfrak{p}}$ is Gorenstein of dimension zero (see [1, Theorem 3]).

Since pure-injective modules are a generalization of injective modules, it is natural to ask whether the tensor product of pure-injective modules is pure-injective. In this paper we deal with this question.

## 2 The Results

A result of Warfield assures us that if $S$ is a maximal immediate extension of a valuation ring $R$ and $M$ is a finitely generated $R$-module, then $M \otimes_R S$ is pure-injective (see [14, Theorem 6]). Therefore, the tensor product of two not necessarily pure-injective modules can be pure-injective. In this direction, the following two propositions give us sufficient conditions which guarantee that the tensor product of two modules is pure-injective.

We recall that an $R$-module $M$ is called *coherent* if it is finitely generated and every finitely generated submodule of $M$ is finitely presented. A ring $R$ is called *coherent* if $R$ as an $R$-module is coherent.

**Proposition 2.1.** *Let $R$ be a ring, $M$ a finitely presented $R$-module, and $P$ a pure-injective $R$-module. Then $M \otimes_R P$ is a pure-injective $R$-module. In addition, $\mathrm{Tor}_i^R(M, P)$ is a pure-injective $R$-module for every $i \geq 1$ provided $R$ is coherent.*

*Proof.* Since $P$ is pure-injective, it is a direct summand of $\mathrm{Hom}_R(L, E)$, where $L$ is the direct sum of a family of finitely presented $R$-modules and $E$ is an injective $R$-module (see [2, Corollary 2.2]). Therefore, $M \otimes_R P$ is a direct summand of $M \otimes_R \mathrm{Hom}_R(L, E)$ which in turn is isomorphic to $\mathrm{Hom}_R(\mathrm{Hom}_R(M, L), E)$ since $M$ is finitely presented. The injectivity of $E$ now implies that $\mathrm{Hom}_R(\mathrm{Hom}_R(M, L), E)$ is pure-injective and therefore $M \otimes_R P$ is pure-injective.

Now suppose $R$ is coherent and $i \geq 1$ is arbitrary. By the argument in the beginning of the proof, we conclude that $\mathrm{Tor}_i^R(M, P)$ is a direct summand of $\mathrm{Tor}_i^R(M, \mathrm{Hom}_R(L, E))$ which in turn is isomorphic to $\mathrm{Hom}_R(\mathrm{Ext}_R^i(M, L), E)$ by [6, Theorem 2.6.6]. The injectivity of $E$ now implies that $\mathrm{Hom}_R(\mathrm{Ext}_R^i(M, L), E)$ is pure-injective and therefore $\mathrm{Tor}_i^R(M, P)$ is pure-injective as required. □

*Remark 2.2.* Let $R$ be a ring and let $M, P$ be $R$-modules, where $P$ is assumed to be pure-injective. Then it is known that $\mathrm{Ext}_R^i(M, P)$ is a pure-injective $R$-module for every $i \geq 0$ (see [14, Proposition 7 and Theorem 2]). By the argument in the beginning of the proof of Proposition 2.1, we may also deduce this result in a different way. In fact, suppose $i \geq 0$ is arbitrary. By the argument we conclude that $\mathrm{Ext}_R^i(M, P)$ is a direct summand of $\mathrm{Ext}_R^i(M, \mathrm{Hom}_R(L, E))$ which in turn is isomorphic to $\mathrm{Hom}_R(\mathrm{Tor}_i^R(M, L), E)$ by [6, Theorem 2.6.6]. The injectivity of $E$ now implies that $\mathrm{Hom}_R(\mathrm{Tor}_i^R(M, L), E)$ is pure-injective and therefore $\mathrm{Ext}_R^i(M, P)$ is pure-injective as required.

We recall that if $M$ is a module over a Noetherian ring $R$ and $0 \to M \to E^0 \to \cdots \to E^n \to E^{n+1} \to \cdots$ is a minimal injective resolution of $R$, where $E^i \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} \mu^i(\mathfrak{p}, M) \mathrm{E}(R/\mathfrak{p})$, then $\mu^i(\mathfrak{p}, M)$ is called the $i$-th *Bass number* of $M$ with respect to $\mathfrak{p}$. It is well known that these numbers are well defined.

**Proposition 2.3.** *Let $R$ be a Noetherian ring. Suppose $E_1$ and $E_2$ are injective $R$-modules such that the* 0*-th Bass numbers of $E_1$ and $E_2$ are finite. If $E_1$ and $E_2$ have only a finite number of associated primes, then $\mathrm{Tor}_i^R(E_1, E_2)$ is a pure-injective $R$-module for every $i \geq 0$.*

*Proof.* First, claim that for any $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(R)$, the $R$-module $\mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{p}), \mathrm{E}(R/\mathfrak{q}))$ is pure-injective for every $i \geq 0$. In order to show this, suppose $i \geq 0$ is arbitrary. If $\mathfrak{p} \neq \mathfrak{q}$, then by [3, Lemma 2.1], we have $\mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{p}), \mathrm{E}(R/\mathfrak{q})) = 0$ which is obviously pure-injective. Hence, we suppose $\mathfrak{p} = \mathfrak{q}$. It is well known that $\mathrm{E}(R/\mathfrak{p})$ is isomorphic to $\mathrm{E}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$ as an $R_\mathfrak{p}$-module and so we obtain

$$\begin{aligned} \mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{p}), \mathrm{E}(R/\mathfrak{p})) &\cong \mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{p}), R_\mathfrak{p} \otimes_{R_\mathfrak{p}} \mathrm{E}(R/\mathfrak{p})) \\ &\cong \mathrm{Tor}_i^{R_\mathfrak{p}}(\mathrm{E}(R/\mathfrak{p}), \mathrm{E}(R/\mathfrak{p})). \end{aligned}$$

On the other hand, if $\varphi : R \to S$ is a homomorphism of rings and $M$ is a pure-injective $S$-module, then $M$ is a pure-injective $R$-module (see [12, Lemma 4.1]). Hence, it is enough to show that if $(R, \mathfrak{m})$ is a Noetherian local ring, then

$$\mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{m}), \mathrm{E}(R/\mathfrak{m}))$$

is pure-injective. But by [3, Lemma 3.3], $\mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{m}), \mathrm{E}(R/\mathfrak{m}))$ is Artinian which implies that it is pure-injective (see [12, Corollary 4.2]). Thus, the claim holds.

Now suppose $i \geq 0$ is arbitrary. Since $E_1$ and $E_2$ are injective, we have $E_1 \cong \bigoplus_{\mathfrak{p}_1 \in \mathrm{Spec}(R)} \mu(\mathfrak{p}_1, E_1)\mathrm{E}(R/\mathfrak{p}_1)$ and $E_2 \cong \bigoplus_{\mathfrak{p}_2 \in \mathrm{Spec}(R)} \mu(\mathfrak{p}_2, E_2)\mathrm{E}(R/\mathfrak{p}_2)$. Note that if $\mathfrak{p}_1 \neq \mathfrak{p}_2$, then by [8, Lemma 2.1], we have $\mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{p}_1), \mathrm{E}(R/\mathfrak{p}_2)) = 0$ and therefore $\mathrm{Tor}_i^R(E_1, E_2) \cong \bigoplus_{\mathfrak{p} \in \mathrm{Spec}(R)} \mu(\mathfrak{p}, E_1)\mu(\mathfrak{p}, E_2)\mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{p}), \mathrm{E}(R/\mathfrak{p}))$. Now the claim implies that $\mathrm{Tor}_i^R(E_1, E_2)$ is a pure-injective $R$-module. □

*Remark 2.4.* It is known that if $R$ is a Gorenstein ring, then for any $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathrm{Spec}(R)$, the $R$-module $\mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{p}), \mathrm{E}(R/\mathfrak{q}))$ is injective (see [3, Theorem 4.1]). This is not the case whenever $R$ is not Gorenstein, but by the claim in the beginning of the proof of Proposition 2.3, we may deduce that $\mathrm{Tor}_i^R(\mathrm{E}(R/\mathfrak{p}), \mathrm{E}(R/\mathfrak{q}))$ is pure-injective.

There are examples in the literature showing that the tensor product of two pure-injective modules need not be pure-injective in general. In this direction, the following two theorems give us conditions which guarantee that the tensor product of two pure-injective modules is pure-injective. We recall that an $R$-module $M$ is called *$\Sigma$-pure-injective* if all direct sums of copies of $M$ are pure-injective. A ring $R$ is called *quasi-Frobenius* if $R$ is Artinian and self-injective. A ring $R$ is called *pure-semisimple* if every $R$-module is a direct sum of finitely generated modules.

**Theorem 2.5.** *Let $R$ be a ring that is either $\Sigma$-pure-injective or quasi-Frobenius. Then the tensor product of pure-injective modules is pure-injective if and only if $R$ is pure-semisimple.*

*Proof.* First, suppose the tensor product of pure-injective modules is pure-injective. Let $M$ be a pure-injective $R$-module. Then since $R$ is $\Sigma$-pure-injective, $R^{(I)}$ is pure-injective. Now $R^{(I)} \otimes_R M \cong M^{(I)}$ and so $M^{(I)}$ is pure-injective. Therefore, by [13, Corollary 11.3], it follows that $R$ is pure-semisimple.

Conversely, if $R$ is pure-semisimple, then every module is pure-injective. Therefore, the tensor product of pure-injective modules is also pure-injective. □

**Theorem 2.6.** *Let $R$ be a ring and let $P, Q$ be Artinian $R$-modules. Then $P \otimes_R Q$ is a pure-injective $R$-module.*

*Proof.* By [4, Proposition 6.1], $P \otimes_R Q$ is a module of finite length and so is Artinian. By [12, Corollary 4.2], $P \otimes_R Q$ is pure-injective. □

## 3 Appendix

Here we consider the notion of linear compactness in order to add to our information about the tensor product of two pure-injective modules. This notion was first introduced by Lefschetz [10] in 1942 for vector spaces and then was continued by Zelinsky [16] and others in the 1950's for not necessarily commutative rings and modules. Let us first recall the concept of linearly compact modules by using the terminology of Macdonald [11].

Suppose $R$ is a *topological ring*, that is, a ring with a topology such that $(R, +)$ is a topological group and the multiplication is continuous. Also, suppose $M$ is a topological $R$-module, by which we mean a module equipped with a topology making it a topological group such that the multiplications by elements of $R$ are continuous. A *nucleus* of $M$ is a neighborhood of the zero element of $M$ and a *nuclear base* of $M$ is a base for the nuclei of $M$. The $R$-module $M$ is called *linearly topologized* if $M$ has a nuclear base consisting of submodules. A Hausdorff linearly topologized $R$-module $M$ is called *linearly compact* if whenever $\{x_i + M_i \mid i \in I\}$ is a family of closed cosets of $M$ in such a way that any finite number of them have a nonempty intersection, then $\bigcap_{i\in I}(x_i + M_i) \neq \emptyset$. We are often interested in the case where $R$ or $M$ or both carry the discrete topology. A linearly compact module in the discrete topology is called a *discrete linearly compact* module. Note that in this case, all topological assumptions may be omitted. In fact, the $R$-module $M$ is discrete linearly compact if whenever $\{x_i + M_i \mid i \in I\}$ is a family of cosets of $M$ in such a way that any finite number of them have a nonempty intersection, then $\bigcap_{i\in I}(x_i + M_i) \neq \emptyset$. Equivalently, if every system of congruences $x \equiv x_i \pmod{M_i}$ has a solution $x$ whenever it has a solution for every finite subsystem.

*Remark 3.1.* There are various definitions for linearly compact modules in the literature. For example, in the definition of linear compactness, some authors assumed that the topological module $M$ carry an arbitrary topology, while the others assumed that the topology of $M$ is linear. Even linearly compact modules with respect to the discrete topology are called linearly compact by some authors. In this direction, the results in the literature may not agree and thus such results must be carefully applied with regard to various definitions of linear compactness.

We recall that an $R$-module $M$ is *finitely embedded* if $M$ is isomorphic to a submodule of the injective envelope of the direct sum of finitely many simple modules. A ring is called a *classical ring* if the finitely embedded modules are discrete linearly compact. Note that there are infinitely many rings which are classical rings. For example, Noetherian rings, maximal valuation rings, and von Neuman regular rings are all classical rings.

It is a fact that every linearly compact module with respect to some topology is pure-injective. This may be easily obtained by combining Proposition 9 together with Theorem 2 in Warfield [14]. Now let us assume that $R$ is a classical ring. In this case, we claim that the converse of the above fact is also true. In fact, we can prove that every pure-injective module is linearly compact with respect to some topology. In order to prove the claim, let $M$ be a pure-injective $R$-module. By [2, Corollary 2.6], the $R$-module $M$ is isomorphic to a direct summand of the direct product of a family of finitely embedded R-modules. Therefore, there are a family $\{L_i \mid i \in I\}$ of finitely embedded $R$-modules and an $R$-module $M'$ such that $\prod_{i\in I} L_i \cong M \oplus M'$. Since $R$ is a classical ring, we obtain that $\prod_{i\in I} L_i$ is linearly compact in the product topology, where we are taking the product of discrete topologies. Therefore, the $R$-module $M \cong \left(\prod_{i\in I} L_i\right)/M'$ is also linearly compact in the quotient topology of $\prod_{i\in I} L_i$ (this topology is not the discrete topology in general). Thus, the $R$-module $M$ is linearly compact in a linear topology and so the claim holds.

Therefore, we observed that over the classical rings, the notion of linear compactness coincides with the notion of pure-injectivity. Thus, any result about the tensor product of two linearly compact modules leads to a result about the tensor product of two pure-injective modules. For example, by [9, Proposition A6(iii)], the tensor product of linearly compact modules is linearly compact (the authors have not been able to verify this claim) and so we are led to the following result: *If $R$ is a classical ring and $P, Q$ are pure-injective $R$-modules, then $P \otimes_R Q$ is a pure-injective $R$-module.*

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