A GENERALIZATION OF THE SWARTZ EQUALITY

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Abstract. For a given $(d-1)$-dimensional simplicial complex Γ, we denote its h-vector by $h(\Gamma) = (h_0(\Gamma), h_1(\Gamma), \ldots, h_d(\Gamma))$ and set $h_{-1}(\Gamma) = 0$. The known Swartz equality implies that if Δ is a $(d-1)$-dimensional Buchsbaum simplicial complex over a field, then for every $0 \le i \le d$, the inequality $ih_i(\Delta) + (d-i+1)h_{i-1}(\Delta) \ge 0$ holds true. In this paper, by using these inequalities, we give a simple proof for a result of Terai (N. Terai, On h-vectors of Buchsbaum Stanley–Reisner rings, *Hokkaido Math. J.* **25** (1996), no. 1, 137–148) on the h-vectors of Buchsbaum simplicial complexes. We then generalize the Swartz equality (E. Swartz, Lower bounds for h-vectors of k-CM, independence, and broken circuit complexes, *SIAM J. Discrete Math.* **18** (2004/05), no. 3, 647–661), which in turn leads to a generalization of the above-mentioned inequalities for Cohen–Macaulay simplicial complexes in co-dimension t.

1. Introduction and preliminaries

Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts and vice versa. One of the fastest developing subfields within algebraic combinatorics is combinatorial commutative algebra. It has evolved into one of the most active areas of mathematics during the past several decades. We refer the reader to the works of Bruns and Herzog [1] and Stanley [3] as general references on the subject.

Let us start with some preliminaries. A *simplicial complex* Δ on the set of vertices $[n] := \{1, \ldots, n\}$ is a collection of subsets of $[n]$ which is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ, the *size* of a face F is defined to be $|F|$ and its *dimension* is defined to be $|F| - 1$. (As usual, for a given finite set X, the number of elements of X is denoted by $|X|$.) The *dimension* of Δ, which is denoted by $\dim \Delta$, is defined to be $d-1$, where $d = \max\{|F| \mid F \in \Delta\}$. A *facet* of Δ is a maximal face of Δ with respect to inclusion. We say that Δ is *pure* if all facets of Δ have the same cardinality. The *link of* Δ *with respect to a face* $F \in \Delta$, denoted by $\mathrm{lk}_\Delta(F)$, is the simplicial complex $\mathrm{lk}_\Delta(F) = \{G \subseteq [n] \setminus F \mid G \cup F \in \Delta\}$.

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One of the connections between combinatorics and commutative algebra is via rings constructed from the combinatorial objects. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$, and let Δ be a simplicial complex on $[n]$. For every subset $F \subseteq [n]$, we set $x_F = \prod_{i \in F} x_i$. The *Stanley–Reisner ideal of* Δ *over* $\mathbb{K}$ is the ideal I_Δ of R which is generated by square-free monomials x_F with $F \notin \Delta$. The *Stanley–Reisner ring of* Δ *over* $\mathbb{K}$, denoted by $\mathbb{K}[\Delta]$, is defined as $\mathbb{K}[\Delta] = R/I_\Delta$. A simplicial complex Δ is called *Cohen–Macaulay over* $\mathbb{K}$ (resp. *Buchsbaum over* $\mathbb{K}$), if its Stanley–Reisner ring $\mathbb{K}[\Delta]$ is a Cohen–Macaulay ring (resp. a Buchsbaum ring).

We now record the numerical data associated with a $(d-1)$-dimensional simplicial complex Δ. Let $f_i(\Delta)$ denote the number of faces of Δ of dimension i. The sequence $f(\Delta) = (f_0(\Delta), f_1(\Delta), \dots, f_{d-1}(\Delta))$ is called the *f-vector* of Δ. Also, to find out more about $f(\Delta)$, we need to study the h-vector of Δ. Although it is seemingly complicated, often it is an elegant way to record the face numbers. Letting $f_{-1}(\Delta) = 1$, we define the *h-vector* $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$ of Δ by the formula

$$\sum_{i=0}^{d} h_i(\Delta) t^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta)(t-1)^{d-i},$$

and for every $i < 0$, we set $h_i(\Delta) = 0$. One can easily check that

$$f_{i-1}(\Delta) = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j(\Delta)$$

and

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}(\Delta).$$

It is well known that

$$H_{\mathbb{K}[\Delta]}(t) = \frac{\sum_{i=0}^{d} h_i(\Delta) t^i}{(1-t)^d},$$

where $H_{\mathbb{K}[\Delta]}(t)$ is the Hilbert series of $\mathbb{K}[\Delta]$ and $d = \dim \Delta + 1$ is its *Krull dimension*.

The study of h-vectors of simplicial complexes has long been a topic of interest both in combinatorics and combinatorial commutative algebra. In this direction, let us state an equality, called the Swartz equality.

Proposition 1.1 ([4], Proposition 2.3)**.** *Let Δ be a pure $(d-1)$-dimensional simplicial complex. Then for every $0 \le i \le d$, we have*

$$ih_i(\Delta) + (d-i+1)h_{i-1}(\Delta) = \sum_{v \in [n]} h_{i-1}(\mathrm{lk}_\Delta(v)).$$

It is well known that Δ is Buchsbaum over a field $\mathbb{K}$ if and only if it is pure and for every non-empty face F of Δ, the link of Δ with respect to F is Cohen–Macaulay over $\mathbb{K}$ (see [3, Theorem 8.1, Page 73]). Also, by a classical result of Stanley, if Δ is Cohen–Macaulay over $\mathbb{K}$ with dimension $d-1$, then for every $0 \le i \le d$, we have

$h_i(\Delta) \geq 0$ (see [3, Theorem 3.3, Page 59]). By combining these facts with Proposition 1.1, the Swartz equality, we conclude the following corollary.

Corollary 1.2. *Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex over a field $\mathbb{K}$. Then for every $0 \leq i \leq d$, we have*

$$ih_i(\Delta) + (d-i+1)h_{i-1}(\Delta) \geq 0.$$

Terai in [5] has proved some inequalities for the h-vectors of Buchsbaum simplicial complexes. In Proposition 2.1, by using Corollary 1.2, we give a simple proof for Terai's result, which shows that this corollary is stronger than his result. This is a motivation for us to see whether we can extend Corollary 1.2 for a larger class of simplicial complexes than that of Buchsbaum. Recently, Haghighi et al. [2] have defined the notion of CM_t simplicial complexes as a generalization of Cohen–Macaulay and Buchsbaum simplicial complexes (see Definition 2.2). This notion is different from that of the so-called t-CM, which may be found in, for example, [4]. The notion of CM_0 simplicial complexes agrees with the notion of Cohen–Macaulay simplicial complexes, and the notion of CM_1 simplicial complexes agrees with that of Buchsbaum simplicial complexes. In Proposition 2.4, we give a generalization of Corollary 1.2 for CM_t simplicial complexes. In order to do this, we need to generalize Proposition 1.1, the Swartz equality, which we state and prove in Theorem 2.3.

2. The results

We start this section by stating the following proposition, which has been already proved by Terai [5]. Here we give a simple proof for this proposition by using Corollary 1.2.

Proposition 2.1 ([5], Theorem 2.1)**.** *Let $\mathbb{K}$ be a field and Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex over $\mathbb{K}$. Then the following inequalities hold true.*

(1) $dh_d(\Delta) + h_{d-1}(\Delta) \geq 0$,

(2) $\binom{d}{2}h_d(\Delta) + (d-1)h_{d-1}(\Delta) + h_{d-2}(\Delta) \geq 0$,

(3) $\binom{d}{3}h_d(\Delta) + \binom{d-1}{2}h_{d-1}(\Delta) + (d-2)h_{d-2}(\Delta) + h_{d-3}(\Delta) \geq 0$,

(4) $\displaystyle\sum_{k=0}^{i}\binom{d+i-k-3}{i-k}h_{d-k}(\Delta) \geq 0, \quad if \quad i \geq 4.$

Proof. By Corollary 1.2, for every $0 \leq i \leq d$, we have $ih_i(\Delta) + (d-i+1)h_{i-1}(\Delta) \geq 0$. This implies that for every $1 \leq i \leq d$, we have

$$\begin{aligned}
0 &\le \frac{1}{i}\sum_{k=1}^{i}\binom{d-k}{i-k}\big((d-k+1)h_{d-k+1}(\Delta)+kh_{d-k}(\Delta)\big)\\
&= \frac{d}{i}\binom{d-1}{i-1}h_d(\Delta)+\frac{1}{i}\sum_{k=1}^{i-1}\left\{k\binom{d-k}{i-k}+(d-k)\binom{d-k-1}{i-k-1}\right\}h_{d-k}(\Delta)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +h_{d-i}(\Delta),
\end{aligned}$$

and so we conclude that

$$(\dagger)\qquad \sum_{k=0}^{i}\binom{d-k}{i-k}h_{d-k}(\Delta)\ge 0.$$

Now the proof of (1), (2) and (3) follows from inequality (†) by substituting $i=1$, $i=2$ and $i=3$ respectively. For proof of (4), by inequality (†), for every $i\ge 4$, we have

$$\begin{aligned}
0 &\le \sum_{j=3}^{i}\sum_{k=0}^{j}\binom{i-3}{i-j}\binom{d-k}{j-k}h_{d-k}(\Delta)\\
&= \sum_{j=3}^{i}\binom{i-3}{i-j}\binom{d}{j}h_d(\Delta)+\sum_{j=3}^{i}\binom{i-3}{i-j}\binom{d-1}{j-1}h_{d-1}(\Delta)\\
&\qquad +\sum_{j=3}^{i}\binom{i-3}{i-j}\binom{d-2}{j-2}h_{d-2}(\Delta)+\sum_{k=3}^{i}\sum_{j=k}^{i}\binom{i-3}{i-j}\binom{d-k}{j-k}h_{d-k}(\Delta)\\
&= \binom{d+i-3}{i}h_d(\Delta)+\binom{d+i-4}{i-1}h_{d-1}(\Delta)+\binom{d+i-5}{i-2}h_{d-2}(\Delta)\\
&\qquad +\sum_{k=3}^{i}\binom{d+i-k-3}{i-k}h_{d-k}(\Delta)\\
&= \sum_{k=0}^{i}\binom{d+i-k-3}{i-k}h_{d-k}(\Delta),
\end{aligned}$$

as required. □

Let us define the notion of CM_t simplicial complexes.

Definition 2.2. Let t be a non-negative integer, $\mathbb{K}$ a field and Δ a pure simplicial complex. We say that Δ is a CM_t *simplicial complex over* $\mathbb{K}$ if for every $F\in\Delta$ with $|F|\ge t$, the link of Δ with respect to F is Cohen–Macaulay over $\mathbb{K}$.

We now generalize Corollary 1.2 for CM_t complexes. In order to do this, we state and prove the following theorem, which is a generalization of Proposition 1.1, the Swartz equality.

Theorem 2.3. *Let Δ be a pure $(d-1)$-dimensional simplicial complex. Then for every $1 \le t \le d$ and every $0 \le i \le d$, we have*

$$\sum_{\substack{F \in \Delta \\ |F| = t}} h_{i-t}(\mathrm{lk}_\Delta(F)) = \sum_{k=0}^{t} \binom{i-k}{t-k}\binom{d-i+k}{k} h_{i-k}(\Delta).$$

Proof. We prove the theorem by induction on t. The case $t=1$ follows from Proposition 1.1. We now assume that $t \ge 2$. Suppose that the assertion is true for $t-1$. Therefore, we have

$$\sum_{\substack{F \in \Delta \\ |F| = t-1}} h_{i-t+1}(\mathrm{lk}_\Delta(F)) = \sum_{k=0}^{t-1} \binom{i-k}{t-k-1}\binom{d-i+k}{k} h_{i-k}(\Delta).$$

By using Proposition 1.1, we conclude that

$$\begin{aligned}
\sum_{\substack{F \in \Delta \\ |F| = t}} h_{i-t}(\mathrm{lk}_\Delta(F)) &= \frac{1}{t} \sum_{\substack{F \in \Delta \\ |F| = t-1}} \sum_{u \in \mathrm{lk}_\Delta(F)} h_{i-t}(\mathrm{lk}_{\mathrm{lk}_\Delta(F)}(u)) \\
&= \frac{1}{t} \sum_{\substack{F \in \Delta \\ |F| = t-1}} \Big\{ (i-t+1) h_{i-t+1}(\mathrm{lk}_\Delta(F)) \\
&\qquad + (d-t+1-i+t) h_{i-t}(\mathrm{lk}_\Delta(F)) \Big\} \\
&= \frac{1}{t}(i-t+1) \sum_{\substack{F \in \Delta \\ |F| = t-1}} h_{i-t+1}(\mathrm{lk}_\Delta(F)) \\
&\qquad + \frac{1}{t}(d-i+1) \sum_{\substack{F \in \Delta \\ |F| = t-1}} h_{i-t}(\mathrm{lk}_\Delta(F)).
\end{aligned}$$

Now, the induction hypothesis implies that

$$
\begin{aligned}
\sum_{\substack{F \in \Delta \\ |F| = t}} h_{i-t}(\mathrm{lk}_\Delta(F)) &= \frac{1}{t}(i-t+1)\sum_{k=0}^{t-1}\binom{i-k}{t-k-1}\binom{d-i+k}{k}h_{i-k}(\Delta) \\
&\quad + \frac{1}{t}(d-i+1)\sum_{k=0}^{t-1}\binom{i-k-1}{t-k-1}\binom{d-i+k+1}{k}h_{i-k-1}(\Delta) \\
&= \binom{i}{t}h_i(\Delta) + \sum_{k=1}^{t-1}\left\{\frac{1}{t}(i-t+1)\binom{i-k}{t-k-1}\binom{d-i+k}{k}\right. \\
&\quad \left. + \frac{1}{t}(d-i+1)\binom{i-k}{t-k}\binom{d-i+k}{k-1}\right\}h_{i-k}(\Delta) \\
&\quad + \binom{d-i+t}{t}h_{i-t}(\Delta) \\
&= \binom{i}{t}h_i(\Delta) + \sum_{k=1}^{t-1}\binom{i-k}{t-k}\binom{d-i+k}{k}h_{i-k}(\Delta) \\
&\quad + \binom{d-i+t}{t}h_{i-t}(\Delta) \\
&= \sum_{k=0}^{t}\binom{i-k}{t-k}\binom{d-i+k}{k}h_{i-k}(\Delta),
\end{aligned}
$$

as required. □

Since every component of the h-vector of a Cohen–Macaulay simplicial complex over a field $\mathbb{K}$ is non-negative, Theorem 2.3 implies the following generalization of Corollary 1.2.

Proposition 2.4. *Let t be a non-negative integer and $\mathbb{K}$ a field. Let Δ be a $(d-1)$-dimensional CM_t simplicial complex over $\mathbb{K}$. Then for every $0 \leq i \leq d$, we have*

$$\sum_{k=0}^{t}\binom{i-k}{t-k}\binom{d-i+k}{k}h_{i-k}(\Delta) \geq 0.$$

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