COHEN–MACAULAYNESS AND LIMIT BEHAVIOR OF DEPTH FOR POWERS OF COVER IDEALS

A. CONSTANTINESCU, M. R. POURNAKI, S. A. SEYED FAKHARI, N. TERAI, AND S. YASSEMI

Dedicated with gratitude to our friend Marco Fontana on the occasion of his 65th birthday

Abstract. Let $\mathbb{K}$ be a field, and let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over $\mathbb{K}$ in n indeterminates $x_1, \dots, x_n$. Let G be a graph with vertex-set $\{x_1, \dots, x_n\}$, and let J be the cover ideal of G in R. For a given positive integer k, we denote the kth symbolic power and the kth bracket power of J by $J^{(k)}$ and $J^{[k]}$, respectively. In this paper, we give necessary and sufficient conditions for R/J^k, $R/J^{(k)}$, and $R/J^{[k]}$ to be Cohen–Macaulay. We also study the limit behavior of the depths of these rings.

1. Introduction

Let R be a commutative ring with identity, and let I be an ideal of R. For a given positive integer k, the study of the ring R/I^k has long been a topic of interest in commutative algebra, and there are many questions which we can formulate. For example, one can ask the following question: *When is R/I^k a Cohen–Macaulay ring*? The study of the depth of R/I^k may help us to understand the initial behavior of R/I^k which is more mysterious. Therefore, it is natural to ask: *What is the limit behavior of the numerical function $f(k) = \operatorname{depth}(R/I^k)$*? There are many papers in the literature which deal with these kinds of questions. Regarding the above questions, note that there are examples of graded ideals such that R/I is Cohen–Macaulay, while R/I^2 is not Cohen–Macaulay. In these cases, $\operatorname{depth}(R/I) < \operatorname{depth}(R/I^2)$ holds true (see, for example, [19]). Let R be either a Noetherian local ring with maximal ideal $\mathfrak{m}$, or a standard graded $\mathbb{K}$-algebra with graded maximal ideal $\mathfrak{m}$, where $\mathbb{K}$ is any field. Let I be a proper ideal of R, which we assume to be graded if R is standard graded. It is clear that the numerical function $f(k) = \operatorname{depth}(R/I^k)$ is bounded by $d = \dim R$. A classical result of Burch [7] improves this fact and says that

$$\min_{k\geq 0} \operatorname{depth}(R/I^k) \leq d - \ell(I),$$

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where $\ell(I)$ denotes the analytic spread of I, that is, the dimension of $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$, where $\mathcal{R}(I)$ is the Rees ring of I. On the other hand, by a theorem of Brodmann [5] the value of $\operatorname{depth}(R/I^k)$ is constant for $k \gg 0$. This constant value is called the limit depth of I and is denoted by $\lim_{k\to\infty} \operatorname{depth}(R/I^k)$. Brodmann improved the Burch inequality by showing that

$$\lim_{k\to\infty} \operatorname{depth}(R/I^k) \leq d - \ell(I).$$

In 1983, Eisenbud and Huneke [12] showed that equality holds if the associated graded ring $\operatorname{gr}_I(R)$ is Cohen–Macaulay. This is, for example, the case if R and $\mathcal{R}(I)$ are Cohen–Macaulay (see [22]).

Let $\mathbb{K}$ be a field, and let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over $\mathbb{K}$ in n indeterminates $x_1, \ldots, x_n$. Let G be a graph with vertex-set $\{x_1, \ldots, x_n\}$, and let J be the cover ideal of G in R. For a given positive integer k, we denote the kth symbolic power and the kth bracket power of J by $J^{(k)}$ and $J^{[k]}$, respectively. It is known that if R/J^k is a Cohen–Macaulay ring, then we have $J^k = J^{(k)}$ and thus $R/J^{(k)}$ is Cohen–Macaulay. Therefore, in order to answer the first question of this section, it is natural to ask the following question: *When is $R/J^{(k)}$ a Cohen–Macaulay ring*? In this paper, we study the analog of questions mentioned in the above for the rings R/J^k, $R/J^{(k)}$, and $R/J^{[k]}$. In particular, in Theorem 4.3, Theorem 3.1, and Corollary 5.2, we prove that the following statements hold (note that, for a given graph G, the graph $\overline{G}$ denotes the complement of G):

1. For $k \geq 2$, the ring R/J^k is Cohen–Macaulay if and only if G is a complete bipartite graph;
2. The ring $R/J^{(2)}$ is Cohen–Macaulay if and only if every path and cycle with more than 3 vertices in $\overline{G}$ has a chord. Also, for $k \geq 3$, the ring $R/J^{(k)}$ is Cohen–Macaulay if and only if any pair of disjoint edges of G is contained in a cycle of length 4;
3. For $k \geq 1$, the ring $R/J^{[k]}$ is Cohen–Macaulay if and only if $\overline{G}$ is a chordal graph.

2. Preliminaries

In this section, we recall some definitions and notations concerning combinatorics, graphs, and commutative algebra for later use. The reader is referred to [4] and [31] for a fuller treatment of the subject.

2.1. **Notions from combinatorics.** A *simplicial complex* Δ on $[n] := \{1, \ldots, n\}$ is a collection of subsets of $[n]$ which is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ and the *dimension* of a face F is defined to be $|F| - 1$. (As usual, for a given finite set X, the number of elements of X is denoted by $|X|$.) The *dimension* of Δ which is denoted by $\dim \Delta$, is defined to be $d-1$, where $d = \max\{|F| \mid F \in \Delta\}$. A *facet* of Δ is a maximal face of Δ with respect to inclusion. Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m \rangle$. We say that Δ is *pure* if all facets of Δ have the same cardinality. A *nonface* of Δ is a subset F of $[n]$ with $F \notin \Delta$. Let $\mathcal{N}(\Delta)$ denote the set of minimal nonfaces of Δ with respect to inclusion. The *link* of Δ with respect to a face $F \in \Delta$, denoted by $\mathrm{lk}_\Delta(F)$, is the simplicial complex $\mathrm{lk}_\Delta(F) = \{G \subseteq [n] \setminus F \mid G \cup F \in \Delta\}$. We now record the numerical data associated to Δ. Let f_i denote the number of faces of Δ of dimension i. The sequence $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of Δ. Letting $f_{-1} = 1$, the *reduced Euler characteristic* of Δ, denoted by $\widetilde{\chi}(\Delta)$, is defined to be $\widetilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i$.

2.2. **Notions from graph theory.** Throughout the paper, by a graph we mean a finite undirected graph without loops or multiple edges. For a graph G, let $V(G)$ denote the set of vertices of G, and let $E(G)$ denote the set of edges of G.

A *bipartite graph* is one whose vertex-set is partitioned into two (not necessarily nonempty) disjoint subsets in such a way that the two end vertices for each edge lie in distinct partitions. Among bipartite graphs, a *complete bipartite graph* is one in which each vertex is joined to every vertex that is not in the same partition. A *matching* in a graph is a set of edges such that no two different edges share a common vertex. The number of edges in a maximum matching of a graph G is called the *matching number* of G and is denoted by $\alpha'(G)$. A subset S of edges of a graph with no isolated vertices is called an *edge covering* of the graph if every vertex of the graph is an end of some edge in S. The number of edges in a minimum edge covering of a graph G is called *edge covering number* of G and is denoted by $\beta'(G)$.

We recall that a *walk* between two vertices x and y of a graph is a sequence $x = v_0, e_1, v_1, \ldots, e_k, v_k = y$ of vertices and edges of the graph, simply denoted by

$$x = v_0 v_1 \cdots v_{k-1} v_k = y,$$

such that for every i with $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. Also, a *path* between x and y is a walk between x and y without repeated vertices. A *cycle* of a graph is a path such that the start and end vertices are the same. Two cycles are considered the same if they consist of the same vertices and edges. The number of edges (counting repeats) in a walk, path, or a cycle, is called its *length*. A *chord* in a path (resp. cycle) is an edge that joins two nonadjacent vertices of the path (resp. cycle). We say that a graph G is a *chordal graph* if every cycle of length at least four in G has a chord. A graph G is called *connected* if for any vertices x and y of G there is a path between x and y. Otherwise, G is called *disconnected*. The maximal connected subgraphs of G are its *connected components*. Here, maximal means that including any more vertices would yield a disconnected subgraph. Any graph is a *union* of its connected components. If the number of connected components of G is equal to one, then G is, of course, connected.

For a graph G and a nonempty subset S of $V(G)$, the *subgraph induced by S*, denoted $G[S]$, is the maximal subgraph of G with vertex-set S. A subset W of $V(G)$

is called a *vertex cover* of the graph G if every edge of G is incident to at least one vertex of W. A vertex cover W is called a *minimal vertex cover* of G if no proper subset of W is a vertex cover of G. A subset A of $V(G)$ is called an *independent subset* of G if the induced subgraph $G[A]$ contains no edges; that is, there are no edges among the vertices of A. Clearly, $A \subseteq V(G)$ is an independent subset of G if and only if $V(G) \setminus A$ is a vertex cover of G.

2.3. **Notions from commutative algebra.** Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$, and let I be a monomial ideal of R. The *Rees algebra* of I in R is defined to be $\mathcal{R}(I) = \bigoplus_{n=0}^{\infty} I^n = R[It] \subseteq R[t]$. The *analytic spread* of I, denoted by $\ell(I)$, is defined to be the Krull dimension of the fiber cone $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$, where $\mathcal{R}(I)$ is the Rees ring of I and $\mathfrak{m} = (x_1, \dots, x_n)$ is the maximal ideal of R. If I is assumed to be proper, then the notation $\mathrm{gr}_I(R)$ indicates the *associated graded ring* of R with respect to the I-adic filtration of R; that is, $\mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$. The *arithmetical rank* of I, denoted by $\mathrm{ara}\, I$, is defined as follows:

$$\mathrm{ara}\, I = \min\Big\{r \in \mathbb{N} \mid \text{there exist } a_1, \dots, a_r \in I \text{ such that } \sqrt{(a_1, \dots, a_r)} = \sqrt{I}\Big\}.$$

If I is assumed to be squarefree, then the *Alexander dual* of I, denoted by $I^\vee$, is the ideal whose primary components are given by the minimal generators of I; that is, if $I = (x_{1,1} \cdots x_{1,l_1}, \dots, x_{r,1} \cdots x_{r,l_r})$, then $I^\vee = (x_{1,1}, \dots, x_{1,l_1}) \cap \cdots \cap (x_{r,1}, \dots, x_{r,l_r})$.

2.4. **Notions from combinatorial commutative algebra.** Algebraic combinatorics is an area of mathematics that employs methods of abstract algebra in various combinatorial contexts. One of the fastest developing subfields within algebraic combinatorics is combinatorial commutative algebra. It has evolved into one of the most active areas of mathematics during the last several decades. We refer the reader to the books by Stanley [31], Bruns and Herzog [6], as well as Miller and Sturmfels [24] as general references in the subject.

One of the connections between the combinatorics and commutative algebra is via rings constructed from the combinatorial objects. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$, and let Δ be a simplicial complex on $[n]$. For every subset $F \subseteq [n]$, we set $x_F = \prod_{i \in F} x_i$. The *facet ideal* of Δ is the ideal $I(\Delta)$ of R which is generated by those squarefree monomials x_F with $F \in \mathcal{F}(\Delta)$. Thus if $\Delta = \langle F_1, \dots, F_m \rangle$, then $I(\Delta) = (x_{F_1}, \dots, x_{F_m})$. The *Stanley–Reisner ideal* of Δ is the ideal I_Δ of S which is generated by those squarefree monomials x_F with $F \notin \Delta$. In other words, $I_\Delta = \big(x_F \mid F \in \mathcal{N}(\Delta)\big)$. The *Stanley–Reisner ring* of Δ, denoted by $\mathbb{K}[\Delta]$, is defined to be $\mathbb{K}[\Delta] = R/I_\Delta$. The simplicial complex Δ is called *Cohen–Macaulay* (resp. *Gorenstein*) if the ring $\mathbb{K}[\Delta]$ is Cohen–Macaulay (resp. Gorenstein). Let G be a graph with $V(G) = \{x_1, \dots, x_n\}$. Then by identifying the vertices of G with the variables in the ring R, we can associate a squarefree quadratic monomial ideal $I(G)$ of R to the graph G. The ideal $I(G)$ is defined by $I(G) = \big(x_i x_j \mid \{x_i, x_j\} \in E(G)\big)$ and is called the *edge ideal* of G in R. The edge ideal, which was first introduced by Villarreal in the 1990s, is an algebraic object whose invariants can be related to the properties of G and vice versa (see [35] as a more accessible reference). The edge ideal of a hypergraph and of a clutter can be defined in a similar way (see [11, 17, 18]). Many authors have been interested in using the edge ideal construction to build a dictionary between the fields of graph theory and commutative algebra. For general references we refer the reader to [30] and [35].

We continue this part by recalling polarization of a monomial ideal which is a deformation that assigns to an arbitrary monomial ideal a squarefree monomial ideal in a new set of variables. Let I be a monomial ideal of $R = \mathbb{K}[x_1, \ldots, x_n]$ with minimal generators $u_1, \ldots, u_m$, where $u_i = \prod_{j=1}^{n} x_j^{a_{ij}}$, $1 \leq i \leq m$. For every j with $1 \leq j \leq n$, let $a_j = \max\{a_{ij} \mid 1 \leq i \leq m\}$, and suppose that

$$T = \mathbb{K}[x_{11}, x_{12}, \ldots, x_{1a_1}, x_{21}, x_{22}, \ldots, x_{2a_2}, \ldots, x_{n1}, x_{n2}, \ldots, x_{na_n}]$$

is a polynomial ring over the field $\mathbb{K}$. Let J be the squarefree monomial ideal of T with minimal generators $v_1, \ldots, v_m$, where $v_i = \prod_{j=1}^{n} \prod_{k=1}^{a_{ij}} x_{jk}$, $1 \leq i \leq m$. The monomial v_i is called the *polarization* of u_i, and the ideal J is called the *polarization* of I.

2.5. **Cover ideals: what we deal with in this paper.** Let G be a graph with vertex-set $V(G) = \{x_1, \ldots, x_n\}$, and let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring over a field $\mathbb{K}$. The Alexander dual of the edge ideal of G in R, i.e., the ideal

$$J(G) = I(G)^{\vee} = \bigcap_{x_i x_j \in E(G)} (x_i, x_j),$$

is called the *cover ideal* of G in R. The reason for this name is due to the well-known fact that the generators of $J(G)$ correspond to vertex covers (see, for example, [20]). Throughout the paper, we simply denote $J(G)$ by J and always "*Let G be a graph, and let J be the cover ideal of G in R*" means that "*Let G be a graph with vertex-set $V(G)$, and let J be the cover ideal of G in the polynomial ring $R = \mathbb{K}[V(G)]$*".

3. Symbolic powers

In this section, we deal with the symbolic powers. Let I be a squarefree monomial ideal in the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$, and suppose that I has the primary decomposition

$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$

where every $\mathfrak{p}_i$ is an ideal of R generated by a subset of the variables of R. Let k be a positive integer. The kth *symbolic power* of I, denoted by $I^{(k)}$, is defined to be

$$I^{(k)} = \mathfrak{p}_1^k \cap \cdots \cap \mathfrak{p}_r^k.$$

Let J be the cover ideal of a graph in the ring R, and let $k \geq 2$ be an integer. We state and prove the following theorem, which gives us a necessary and sufficient condition for $R/J^{(k)}$ to be Cohen–Macaulay in terms of the graph invariants. In the sequel, for a given graph G, the graph $\overline{G}$ denotes the complement of G.

Theorem 3.1. *Let G be a graph, and let J be the cover ideal of G in R.*

(1) *The following statements are equivalent:*

(a) *The ring $R/J^{(2)}$ is Cohen–Macaulay;*
(b) *Every path and cycle with more than 3 vertices in $\overline{G}$ has a chord.*

(2) *Let $k \geq 3$. Then the following statements are equivalent:*
(c) *The ring $R/J^{(k)}$ is Cohen–Macaulay;*
(d) *Any pair of disjoint edges of G is contained in a cycle of length 4.*

Proof. The item (2) is proved in [32, Corollary 5.7]. For proving the item (1), we use polarization. Let J' be the polarization of $J^{(2)}$, which we consider it in the new polynomial ring $T = \mathbb{K}[x_1, \ldots, x_{n'}]$. It is known that the ring $R/J^{(2)}$ is Cohen–Macaulay if and only if the ring T/J' is Cohen–Macaulay (see, for example, [13, Proposition 2.8] or [14]).

We now claim that J' is the cover ideal of a graph. In order to prove the claim, it is enough to show that J' is an unmixed monomial ideal of height two in T. To prove the unmixedness of J', we show that the height of every associated prime ideal of J' is equal to two. Using the fact that polarization preserves the intersection (see [13, Proposition 2.3]), we may deal only with ideals in the form of $I = (x_i, x_j)$ in R. It is clear that R/I is Cohen–Macaulay. Let I' be the polarization of I^2 in the polynomial ring R'. Then R'/I' is Cohen–Macaulay. Since the projective dimension of R/I is equal to two, the projective dimension of T/I' is also equal to two and so $\operatorname{depth}(T/I') = n' - 2$. This implies that $\max\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}(T/I')\} = 2$. Since $\operatorname{ht}(I') = 2$, we conclude that the height of every associated prime ideal of T/I' is equal to two and this proves that I' is unmixed and so J' is also unmixed. On the other hand, polarization preserves the height of an ideal and so we conclude that $\operatorname{ht} J' = \operatorname{ht} J^{(2)} = \operatorname{ht} J = 2$. Thus the claim holds, i.e., there exists a graph, say G', such that the cover ideal of G' is J'. Therefore, the ring T/J' is Cohen–Macaulay if and only if the complement of G', i.e., $\overline{G'}$, is a chordal graph (see [2, Proposition 2.1]).

The above observations imply that, in order to prove (1), it is enough to show that

the graph $\overline{G'}$ is chordal $\Longleftrightarrow$ *every path and cycle with more than 3 vertices in $\overline{G}$ has a chord.*

We complete the proof of theorem by giving a proof for the above statement. Let $I := (x_i, x_j)$, and let I' be the polarization of I^2 in the polynomial ring R'. One can easily show that

$$I' = (x_{i_1}, x_{j_1}) \cap (x_{i_1}, x_{j_2}) \cap (x_{j_1}, x_{i_2})$$

is the irreducible primary decomposition of I'. The above observation shows that the graph G', in fact, is a graph which is obtained from the graph G in such a way that every edge of G, say $x_i x_j$, is replaced by the set $\{x_{i_1}x_{j_1}, x_{i_1}x_{j_2}, x_{j_1}x_{i_2}\}$ of edges.

Proof of ($\Longrightarrow$). Using Theorem 2.6 of [21], we conclude that $R/\sqrt{J^{(2)}} = R/J$ is Cohen–Macaulay, and so by [2, Proposition 2.1], the graph $\overline{G}$ is chordal. Therefore, every cycle with more than 3 vertices in $\overline{G}$ has a chord. Now suppose, on the contrary, that there exists a path P in $\overline{G}$ with at least four vertices and no chord:

$$P: \quad y_1y_2\cdots y_m,$$

where $m \geq 4$. We now consider the cycle

$$C: \quad y_{1_1}y_{2_1}\cdots y_{m_1}y_{m_2}y_{1_2}y_{1_1}$$

in $\overline{G'}$. Since P is a path with no chord in $\overline{G}$, the only chords of C are $y_{m-1_1}y_{m_2}$ and $y_{1_2}y_{2_1}$. Therefore, the cycle

$$C': \quad y_{m-1_1}y_{m_2}y_{1_2}y_{2_1}y_{3_1}\cdots y_{m_{1_1}}$$

has no chord. This implies that $\overline{G'}$ is not chordal, a contradiction. Therefore, every path and cycle with more than 3 vertices in $\overline{G}$ has a chord.

Proof of ($\Longleftarrow$). Suppose, on the contrary, that $\overline{G'}$ is not a chordal graph. Therefore, $\overline{G'}$ has a cycle with at least 4 vertices and no chord, say

$$C: \quad y_1y_2\cdots y_ty_{t+1} = y_1.$$

Since for every $i \neq j$, the vertices x_{i_2} and x_{j_2} are connected in $\overline{G'}$ and also every path and cycle with more than 3 vertices has a chord in $\overline{G}$, we conclude that $t \leq 5$. Hence, one of the following cases occurs.

Case 1: $t = 5$. In this case, C is a cycle in the following form:

$$C: \quad x_{i_1}x_{j_1}x_{k_1}x_{l_2}x_{s_2}x_{i_1}.$$

If $k = l$ (resp. $s = i$), then x_{j_1} is adjacent to x_{l_2} (resp. x_{j_1} is adjacent to x_{s_2}), which is a contradiction, since the cycle C has no chord. Therefore, $k \neq l$ and $s \neq i$. Since $x_{s_2}x_{i_1}$ is an edge of $\overline{G'}$, x_i is not adjacent to x_s in G. Also, $x_{i_1}x_{j_1}$ and $x_{j_1}x_{k_1}$ are edges of $\overline{G'}$ and so x_i is not adjacent to x_j, and x_j is not adjacent to x_k in G. Since C has no chord, x_k is adjacent to x_s, x_j is adjacent to x_s, and x_i is adjacent to x_k in G. This implies that $x_sx_ix_jx_k$ is a path in $\overline{G}$ with no chord, a contradiction.

Case 2: $t = 4$. In this case, C is a cycle in one of the following forms:

$$C: \quad x_{i_1}x_{j_1}x_{k_1}x_{l_2}x_{i_1} \quad \text{or} \quad x_{i_1}x_{j_1}x_{k_2}x_{l_2}x_{i_1}.$$

In the first form, similar to the Case 1, we conclude that $k \neq l$ and $i \neq l$. This implies that $x_ix_jx_kx_lx_i$ is a cycle in $\overline{G}$ with no chord, a contradiction. In the second form, again similar to the Case 1, we conclude that $k \neq j$ and $i \neq l$. This implies that $x_lx_ix_jx_k$ is a path in $\overline{G}$ with no chord, again a contradiction.

Since both cases lead to a contradiction, we conclude that $\overline{G'}$ is a chordal graph. This completes the proof of the theorem. □

The following theorem gives us the limit behavior of $R/J^{(k)}$, where J is the cover ideal of a bipartite graph in R.

Theorem 3.2. *Let G be a bipartite graph, and let J be the cover ideal of G in R. Also, let k be a positive integer. Then we have* $\operatorname{depth}(R/J^{(k+1)}) \leq \operatorname{depth}(R/J^{(k)})$.

Proof. Since G is a bipartite graph, we may partition the vertex-set of G into two disjoint subsets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$. We have

$$J = \bigcap_{x_i y_j \in E(G)} (x_i, y_j),$$

and so we conclude that

$$J^{(k+1)} = \bigcap_{x_i y_j \in E(G)} (x_i, y_j)^{k+1}.$$

We know that polarization commutes with the intersection (see [13, Proposition 2.3]). Therefore, the polarization of $J^{(k+1)}$ is an ideal in the polynomial ring

$$S = \mathbb{K}[x_{11}, \dots, x_{n(k+1)}, y_{11}, \dots, y_{m(k+1)}],$$

and it is equal to

$$\big(J^{(k+1)}\big)_{\mathrm{pol}} = \bigcap_{x_i y_j \in E(G)} \big((x_i, y_j)^{k+1}\big)_{\mathrm{pol}}.$$

If we replace x_{i2} and y_{jk} by 1 for every $1 \leq i \leq n$ and $1 \leq j \leq m$, then the resulting ideal, say I, is in fact, the localization of $\big(J^{(k+1)}\big)_{\mathrm{pol}}$ at the prime ideal

$$(x_{is}, y_{jt} \mid 1 \leq i \leq n,\ 1 \leq j \leq m,\ 1 \leq s, t \leq k+1,\ s \neq 2,\ t \neq k).$$

Now, I is an ideal in a ring, say S', and it is equal to the polarization of $J^{(k)}$. We know that polarization preserves projective dimension and so

$$\begin{aligned} \mathrm{pd}_R(R/J^{(k+1)}) &= \mathrm{pd}_S\big(S/\big(J^{(k+1)}\big)_{\mathrm{pol}}\big) \\ &\geq \mathrm{pd}_{S'}(S'/I) \\ &= \mathrm{pd}_R(R/J^{(k)}). \end{aligned}$$

Therefore, by using Auslander–Buchsbaum formula, we conclude that $\mathrm{depth}(R/J^{(k+1)}) \leq \mathrm{depth}(R/J^{(k)})$, as required. □

We continue the paper with the definition of the size of a monomial ideal (see [23]). Then we state and prove a lemma which is useful to the proof of Theorem 3.5.

Definition 3.3. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$, and let I be a monomial ideal of R. Let $I = \bigcap_{j=1}^{r} Q_j$ be an irredundant primary decomposition of I, where Q_j $(1 \leq j \leq r)$ is a monomial ideal of R. Let h be the height of $\sum_{j=1}^{r} Q_j$, and denote by v the minimum number t such that there exist $1 \leq j_1, \dots, j_t \leq r$ with

$$\sqrt{\sum_{i=1}^{t} Q_{j_i}} = \sqrt{\sum_{j=1}^{r} Q_j}.$$

Then the *size* of I is defined to be $v + n - h - 1$.

Lemma 3.4. *Let G be a graph, and let J be the cover ideal of G in R. Also, let k be a positive integer. Then the size of $J^{(k)}$ is equal to $\beta'(G) - 1$.*

Proof. We know that

$$J^{(k)} = \bigcap_{x_i x_j \in E(G)} (x_i, x_j)^k.$$

Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the maximal ideal of R. Note that the height of $\mathfrak{m}$ is equal to $n = |V(G)|$. On the other hand, $\sum_{x_i x_j \in E(G)} (x_i, x_j)^k$ is an $\mathfrak{m}$-primary ideal of R. Since for every subset S of $E(G)$ the equality

$$\sqrt{\sum_{x_i x_j \in S} (x_i, x_j)^k} = \sqrt{\sum_{x_i x_j \in E(G)} (x_i, x_j)^k} = \mathfrak{m}$$

holds true if and only if S is an edge covering of G, we conclude that the size of $J^{(k)}$ is equal to $\beta'(G) - 1$, as required. □

We are now ready to state and prove the following theorem.

Theorem 3.5. *Let G be a graph, and let J be the cover ideal of G in R. Also, let k be a positive integer. Then the following statements hold:*

(a) $\beta'(G) - 1 \leq \min_{k \geq 0} \operatorname{depth}(R/J^{(k)}) \leq n - \operatorname{ara} J$;

(b) $\operatorname{ara} J \leq \alpha'(G) + 1$.

Proof. For proof of (a), by Proposition 3 of [23], for every squarefree monomial ideal I of R, we have the following inequalities:

$$n - \operatorname{depth}(R/I) \leq \operatorname{ara} I \leq n - \min_{k \geq 0} \operatorname{depth}(R/I^{(k)}).$$

This implies that

$$\min_{k \geq 0} \operatorname{depth}(R/J^{(k)}) \leq n - \operatorname{ara} J.$$

On the other hand, by Proposition 2 of [23], for every monomial ideal I of R, the quantity $\operatorname{depth}(R/I)$ is grater than or equal to the size of I. Therefore, by Lemma 3.4, we have

$$\min_{k \geq 0} \operatorname{depth}(R/J^{(k)}) \geq \beta'(G) - 1.$$

For proof of (b), note that by Theorem 7.2 of [4], for every graph G we have $\alpha'(G) + \beta'(G) = n$. Therefore, by part (a), we conclude that $\operatorname{ara} J \leq \alpha'(G) + 1$, as required. □

4. Ordinary powers

Recently, there are many works on Cohen–Macaulayness of powers of squarefree monomial ideals (see, for example, [10, 25, 26, 27, 28, 32, 33, 34]). In this section, we focus on Cohen–Macaulayness of powers of the cover ideals of graphs.

We first prove the following lemma which is useful for the proof of subsequent theorem.

Lemma 4.1. *Let Δ be a simplicial complex on $[n]$, and let $I_\Delta \subseteq R = \mathbb{K}[x_1, \dots, x_n]$ denote the Stanley–Reisner ideal of Δ. Also, let $k \geq 2$ be an integer. Suppose R/I_Δ^k is Cohen–Macaulay for every field $\mathbb{K}$. Then Δ is Gorenstein for every field $\mathbb{K}$.*

Proof. Set $d = \dim \Delta + 1$. First, assume that $d = 1$. Then one can take a complete graph G such that $I(G) = I_\Delta$. Since $R/I(G)^k$ is Cohen–Macaulay, $I(G)^{(k)} = I(G)^k$ and so G does not contain any triangle by [27]. Therefore, $|V(\Delta)| = |V(G)| \leq 2$ which implies that Δ is Gorenstein.

Second, assume that $d \geq 2$. We may assume Δ is not a cone, i.e., every x_i is a zero-divisor in R/I_Δ. Since R/I_Δ^k is Cohen–Macaulay, R/I_Δ is Cohen–Macaulay by, for example, [21, Introduction]. Let F be a face of Δ such that $\dim \Gamma = 1$, where we set $\Gamma = \mathrm{lk}_\Delta(F)$. Since R/I_Γ^k is Cohen–Macaulay, either Γ is a cycle of length three, four, or five, or a path with two or three vertices by [25, Corollaries 3.4 and 3.5]. Therefore, Γ is Gorenstein.

Now suppose that the characteristic of $\mathbb{K}$ is two. By [31, Chapter II, Theorem 5.1], Δ is Gorenstein over $\mathbb{K}$. Then the reduced Euler characteristic $\widetilde{\chi}(\Delta)$ of Δ is $(-1)^{d-1}$.

Now, let $\mathbb{K}$ be an arbitrary field. We have $\widetilde{\chi}(\Delta) = (-1)^{d-1}$, since $\widetilde{\chi}(\Delta)$ is independent of $\mathbb{K}$. Therefore, Δ is Gorenstein over $\mathbb{K}$ by [31, Chapter II, Theorem 5.1]. □

Remark 4.2. The above proof is a simplified generalized version of the proof in [28]. In [32] another proof is given for a similar statement.

We now prove the following theorem which gives us a necessary and sufficient condition for R/J^k to be Cohen–Macaulay, where J is the cover ideal of a graph.

Theorem 4.3. *Let G be a graph, and let J be the cover ideal of G in R. Also, let $k \geq 2$ be an integer. Then the following statements are equivalent:*

(a) *The ring R/J^k is Cohen–Macaulay;*
(b) *The ring R/J is Gorenstein;*
(c) *The ring R/J is complete intersection;*
(d) *The graph G is complete bipartite.*

Proof. (a $\Longrightarrow$ b): Note that the Cohen–Macaulayness of the ring R/J^k is independent of the base field, since J^k is a monomial ideal with height two. Hence, we are done by Lemma 4.1.

(b $\Longrightarrow$ c): Since the codimension of the ring R/J is equal to two, the Gorenstein property of R/J implies its complete intersection property by [29, Proposition 3].

(c $\Longrightarrow$ d): Since the ring R/J is complete intersection of codimension two, the squarefree monomial ideal J is of the form (m_1, m_2), where $m_1 = x_1x_2\cdots x_s$ and $m_2 = y_1y_2\cdots y_t$ are mutually coprime squarefree monomials. Therefore, G is the complete bipartite graph with the partite sets $\{x_1, x_2, \dots, x_s\}$ and $\{y_1, y_2, \dots, y_t\}$.

(d $\Longrightarrow$ a): Since the graph G is complete bipartite, the ring R/J is complete intersection. Therefore, by [1, 9, 36], R/J^k is Cohen–Macaulay. □

By using Theorem 3.2, the numerical function $f(k) = \operatorname{depth}(R/J^{(k)})$, when J is the cover ideal of a bipartite graph, is nonincreasing. Thus it has a limit when $k \longrightarrow \infty$. In the following, we compute this limit. In order to do this, we first introduce some definitions and remarks.

Definitions and Remarks 4.4. Let G be a graph, and let $M = \{\{a_i, b_i\} \mid 1 \leq i \leq r\}$ be a nonempty matching of G. We say that M is an *ordered matching* of G if the following hold:

(1) $A := \{a_1, \ldots, a_r\} \subseteq V(G)$ is a set of independent vertices of G; and
(2) $a_i b_j \in E(G)$ implies that $i \leq j$.

In this case, the set A is called a *free parameter set* of G and $B = \{b_1, \ldots, b_r\} \subseteq V(G)$ is called a *partner set* of A. The *ordered matching number* of G, denoted by $\nu_o(G)$, is defined to be

$$\nu_o(G) = \max\{|M| \mid M \subseteq E(G) \text{ is an ordered matching of } G\}.$$

The notion of ordered matching was introduced by the first author together with Varbaro in [8]. It was proven in the same paper that the Krull dimension of the symbolic fiber cone of the cover ideal is equal to the ordered matching number $+1$. As we use the ordered matching number for bipartite graphs, it is worth mentioning that this notion extends the notion of graphical dimension introduced by Benedetti and the two above-mentioned authors [3] for bipartite graphs only. In [8] it is shown that for bipartite graphs the graphical dimension $+1$ and the ordered matching number coincide. This means that, if $V(G) = V_1 \cup V_2$ is a partition of the vertex-set, then one can choose $A \subseteq V_1$ and $B \subseteq V_2$.

We are now ready to state and prove the following theorem.

Theorem 4.5. *Let G be a bipartite graph, and let J be the cover ideal of G in R. Also, let k be a positive integer. Then the following statement holds:*

$$\lim_{k\to\infty} \operatorname{depth}(R/J^{(k)}) = \lim_{k\to\infty} \operatorname{depth}(R/J^{k}) = n - 1 - \nu_o(G).$$

Proof. By Corollary 2.6 of [16], we have $J^{(k)} = J^k$. Also, by a classical result of Burch [7], we have

$$\min_{k\geq 0} \operatorname{depth}(R/J^{k}) \leq n - \ell(J),$$

where $\ell(J)$ is the analytic spread of J. Thus we conclude that

$$\lim_{k\to\infty} \operatorname{depth}(R/J^{(k)}) = \lim_{k\to\infty} \operatorname{depth}(R/J^{k}) \leq n - \ell(J) = n - 1 - \nu_o(G),$$

where the last equality holds true by Corollary 2.9 of [8]. By Theorem 4.2 of [20], the Rees algebra $\mathcal{R}(J)$ is Cohen–Macaulay. Therefore, the associated graded ring $\operatorname{gr}_J(R)$ is Cohen–Macaulay, and so by Theorem 1.2 of [19], we have

$$\lim_{k\to\infty} \operatorname{depth}(R/J^{(k)}) = \lim_{k\to\infty} \operatorname{depth}(R/J^{k}) = n - 1 - \nu_o(G),$$

as required. $\square$

The ordered matching number has lower and upper bounds given by some well-known graph invariants. Using these bounds we obtain an easy consequence of Theorem 4.5. First, we introduce two definitions from graph theory.

Definitions 4.6. Let G be a graph. A subset S of $V(G)$ is called a *paired-dominating set* of G if every vertex of G is adjacent to some vertex of S and if the subgraph induced by S has at least one perfect matching. The minimum cardinality of a paired-dominating set is called the *paired-domination number* of G and is denoted by $\gamma_{\mathrm{P}}(G)$.

The following inequalities hold true by Theorem 4.5 and [8, Remark 2.4].

Corollary 4.7. *Let G be a bipartite graph, and let J be the cover ideal of G in R. Also, let k be a positive integer. Then the following statement holds:*

$$n-1-\alpha'(G) \leq \lim_{k\to\infty} \operatorname{depth}(R/J^{(k)}) = \lim_{k\to\infty} \operatorname{depth}(R/J^{k}) \leq n-1-\frac{\gamma_{\mathrm{P}}(G)}{2}.$$

In [8, Proposition 2.10], it is shown that if a graph is a tree, then the ordered matching number and the matching number coincide. This implies the following corollary.

Corollary 4.8. *Let G be a tree, and let J be the cover ideal of G in R. Also, let k be a positive integer. Then the following statement holds:*

$$\lim_{k\to\infty} \operatorname{depth}(R/J^{(k)}) = \lim_{k\to\infty} \operatorname{depth}(R/J^{k}) = n-1-\alpha'(G).$$

5. Bracket powers

In this section, we deal with the bracket powers. Let $I=(f_1,\ldots,f_m)$ be a monomial ideal in the polynomial ring $R=\mathbb{K}[x_1,\ldots,x_n]$ over a field $\mathbb{K}$, and let k be a positive integer. The kth *bracket power* of I, denoted by $I^{[k]}$, is defined to be

$$I^{[k]} = (f_1^k,\ldots,f_m^k).$$

We now introduce the lcm-lattice of a monomial ideal. Let R be as above, and let I be a monomial ideal minimally generated by monomials $m_1,\ldots,m_d$. We denote by L_I the lattice with elements labeled by the least common multiples of $m_1,\ldots,m_d$ ordered by divisibility. In particular, the atoms in L_I are $m_1,\ldots,m_d$, the maximal element is $\operatorname{lcm}(m_1,\ldots,m_d)$, and the minimal element is 1 regarded as the lcm of the empty set. The least common multiple of elements in L_I is their join, i.e., their least common upper bound in the poset L_I. We call L_I the *lcm-lattice* of I. Gasharov, Peeva, and Welker have proven that the lcm-lattice of a monomial ideal determines its free resolution and therefore its projective dimension (see [15, Theorem 3.3]).

We close the paper by the following two results.

Theorem 5.1. *Let $R=\mathbb{K}[x_1,\ldots,x_n]$ be the polynomial ring over a field $\mathbb{K}$, and let I be a monomial ideal of R. Also, let k be a positive integer. Then we have* $\operatorname{depth}(R/I)=\operatorname{depth}(R/I^{[k]})$.

Proof. Note that I and $I^{[k]}$ have the same lcm-lattice and thus the same projective dimension. Therefore, $\mathrm{pd}_R(R/I) = \mathrm{pd}_R(R/I^{[k]})$, and so Auslander–Buchsbaum formula completes the proof. □

Corollary 5.2. *Let G be a graph, and let J be the cover ideal of G in R. Also, let k be a positive integer. Then the following statements are equivalent:*

(a) *The ring $R/J^{[k]}$ is Cohen–Macaulay;*
(b) *The graph $\overline{G}$ is chordal.*

Proof. The ring $R/J^{[k]}$ is Cohen–Macaulay if and only if $\mathrm{depth}(R/J^{[k]}) = n - 2$. Therefore, by using the above theorem, $R/J^{[k]}$ is Cohen–Macaulay if and only if $\mathrm{depth}(R/J) = n - 2$ if and only if $\overline{G}$ is a chordal graph (see [2, Proposition 2.1]). □

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References

[1] R. Achilles, W. Vogel, Über vollständige Durchschnitte in lokalen Ringen, *Math. Nachr.* **89** (1979), 285–298.
[2] S. Ahmad, Cohen–Macaulay intersections, *Arch. Math. (Basel)* **92** (2009), no. 3, 228–236.
[3] B. Benedetti, A. Constantinescu, M. Varbaro, Dimension, depth and zero-divisors of the algebra of basic K-covers of a graph, *Matematiche (Catania)* **63** (2008), no. 2, 117–156.
[4] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer-Verlag, New York, 2008.
[5] M. Brodmann, The asymptotic nature of the analytic spread, *Math. Proc. Cambridge Philos. Soc.* **86** (1979), no. 1, 35–39.
[6] W. Bruns, J. Herzog, *Cohen–Macaulay Rings*, Second Edition, Cambridge University Press, Cambridge, 1998.
[7] L. Burch, Codimension and analytic spread, *Proc. Cambridge Philos. Soc.* **72** (1972), 369–373.
[8] A. Constantinescu, M. Varbaro, Koszulness, Krull dimension, and other properties of graph-related algebras, *J. Algebraic Combin.* **34** (2011), no. 3, 375–400.
[9] R. C. Cowsik, M. V. Nori, On the fibers of blowing up, *J. Indian Math. Soc. (N.S.)* **40** (1976), no. 1-4, 217–222.
[10] M. Crupi, G. Rinaldo, N. Terai, K. I. Yoshida, Effective Cowsik–Nori theorem for edge ideals, *Comm. Algebra* **38** (2010), no. 9, 3347–3357.
[11] L. A. Dupont, R. H. Villarreal, Symbolic Rees algebras, vertex covers and irreducible representations of Rees cones, *Algebra Discrete Math.* **10** (2010), no. 2, 64–86.
[12] D. Eisenbud, C. Huneke, Cohen–Macaulay Rees algebras and their specialization, *J. Algebra* **81** (1983), no. 1, 202–224.
[13] S. Faridi, Monomial ideals via square-free monomial ideals, *Lect. Notes Pure Appl. Math.* **244** (2006), 85–114.
[14] R. Fröberg, A study of graded extremal rings and of monomial rings, *Math. Scand.* **51** (1982), no. 1, 22–34.

[15] V. Gasharov, I. Peeva, V. Welker, The lcm-lattice in monomial resolutions, *Math. Res. Lett.* **6** (1999), no. 5-6, 521–532.

[16] I. Gitler, E. Reyes, R. H. Villarreal, Blowup algebras of ideals of vertex covers of bipartite graphs, *Contemp. Math.* **376** (2005), 273–279.

[17] H. T. Hà, S. Morey, Embedded associated primes of powers of square-free monomial ideals, *J. Pure Appl. Algebra* **214** (2010), no. 4, 301–308.

[18] H. T. Hà, S. Morey, R. H. Villarreal, Cohen–Macaulay admissible clutters, *J. Commut. Algebra* **1** (2009), no. 3, 463–480.

[19] J. Herzog, T. Hibi, The depth of powers of an ideal, *J. Algebra* **291** (2005), no. 2, 534–550.

[20] J. Herzog, T. Hibi, N. V. Trung, Symbolic powers of monomial ideals and vertex cover algebras, *Adv. Math.* **210** (2007), no. 1, 304–322.

[21] J. Herzog, Y. Takayama, N. Terai, On the radical of a monomial ideal, *Arch. Math. (Basel)* **85** (2005), no. 5, 397–408.

[22] C. Huneke, On the associated graded ring of an ideal, *Illinois J. Math.* **26** (1982), no. 1, 121–137.

[23] G. Lyubeznik, On the arithmetical rank of monomial ideals, *J. Algebra* **112** (1988), no. 1, 86–89.

[24] E. Miller, B. Sturmfels, *Combinatorial Commutative Algebra*, Springer-Verlag, New York, 2005.

[25] N. C. Minh, N. V. Trung, Cohen–Macaulayness of powers of two-dimensional squarefree monomial ideals, *J. Algebra* **322** (2009), no. 12, 4219–4227.

[26] N. C. Minh, N. V. Trung, Cohen–Macaulayness of monomial ideals and symbolic powers of Stanley–Reisner ideals, *Adv. Math.* **226** (2011), no. 2, 1285–1306.

[27] G. Rinaldo, N. Terai, K. I. Yoshida, Cohen–Macaulayness for symbolic power ideals of edge ideals, *J. Algebra* **347** (2011), 1–22.

[28] G. Rinaldo, N. Terai, K. I. Yoshida, On the second powers of Stanley–Reisner ideals, *J. Commut. Algebra* **3** (2011), no. 3, 405–430.

[29] J. P. Serre, Sur les modules projectifs, *Seminaire Dubreil*, 1960/61.

[30] A. Simis, W. V. Vasconcelos, R. H. Villarreal, On the ideal theory of graphs, *J. Algebra* **167** (1994), no. 2, 389–416.

[31] R. P. Stanley, *Combinatorics and Commutative Algebra*, Second Edition, Birkhäuser Boston, Inc., Boston, MA, 1996.

[32] N. Terai, N. V. Trung, Cohen–Macaulayness of large powers of Stanley–Reisner ideals, *Adv. Math.* **229** (2012), no. 2, 711–730.

[33] N. V. Trung, T. M. Tuan, Equality of ordinary and symbolic powers of Stanley–Reisner ideals, *J. Algebra* **328** (2011), 77–93.

[34] M. Varbaro, Symbolic powers and matroids, *Proc. Amer. Math. Soc.* **139** (2011), no. 7, 2357–2366.

[35] R. H. Villarreal, *Monomial Algebras*, Monographs and Textbooks in Pure and Applied Mathematics, **238**, Marcel Dekker, Inc., New York, 2001.

[36] R. Waldi, Vollständige Durchschnitte in Cohen–Macaulay-Ringen, *Arch. Math. (Basel)* **31** (1978/79), no. 5, 439–442.

A. Constantinescu, Department of Mathematics, University of Genoa, I-16146 Genoa, Italy.
E-mail address: constant@dima.unige.it

M. R. Pournaki, Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
E-mail address: pournaki@ipm.ir
URL: http://math.ipm.ac.ir/∼pournaki/

S. A. Seyed Fakhari, School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
E-mail address: fakhari@ipm.ir
URL: http://math.ipm.ac.ir/∼fakhari/

N. Terai, Department of Mathematics, Faculty of Culture and Education, Saga University, Saga 840-8502, Japan.
E-mail address: terai@cc.saga-u.ac.jp

S. Yassemi, School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.
E-mail address: yassemi@ipm.ir
URL: http://math.ipm.ac.ir/∼yassemi/