CLASSIFICATION OF RINGS WITH UNIT GRAPHS HAVING DOMINATION NUMBER LESS THAN FOUR

S. KIANI, H. R. MAIMANI, M. R. POURNAKI, AND S. YASSEMI

Dedicated with gratitude to our friend Alberto Facchini
on the occasion of his 60th birthday

Abstract. Let R be a finite commutative ring with nonzero identity. The unit graph of R is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x + y$ is a unit element of R. In this paper, a classification of finite commutative rings with nonzero identity in which their unit graphs have domination number less than four is given.

1. Introduction

Throughout the paper by a graph we mean a finite undirected graph without loops or multiple edges. Also, all rings are finite commutative with nonzero identity and all fields are finite. Therefore, in this paper, we restrict ourselves to the finite case, although some of the results hold true in the infinite case. For undefined terms and concepts, the reader is referred to [12] and [2].

Let G be a graph with vertex set V. A subset D of V is called a *dominating set* of G if every vertex in $V \setminus D$ is adjacent to at least a vertex in D. In other words, D dominates the vertices outside D. A *γ-set* of G is a minimum dominating set of G, that is, a dominating set of G whose cardinality is minimum. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a γ-set of G. The study of domination has long been a topic of interest both in graph theory and complexity theory. It was first considered by Ore who introduced the concept of minimum dominating sets of vertices in a graph. The *dominating set problem* concerns testing whether $\gamma(G) \leq k$ for a given graph G and integer k. The problem is a classical NP-complete decision problem in computational complexity theory (see, for example, [3]). Therefore, it is believed that there is no efficient algorithm that finds a smallest dominating set of a given graph. The first volume of the two-volume book by Haynes, Hedetniemi and Slater [5, 6] provides a comprehensive introduction to "domination in graphs".

2000 *Mathematics Subject Classification.* Primary: 05C75; Secondary: 13M05.
Key words and phrases. Unit graph, Domination number, Total domination number, Finite ring.
The research of H. R. Maimani, M. R. Pournaki, and S. Yassemi was in part supported by a grant from IPM (No. 93050113, No. 93130115, and No. 93130214). The research of M. R. Pournaki was also partially supported by a grant from INSF.

In this paper, we consider graphs which are generated by rings, known as unit graphs. We then give a classification of rings in which their unit graphs have domination number less than four.

2. Preliminaries and the statement of the main result

Let n be a positive integer and $\mathbb{Z}_n$ be the ring of integers modulo n. Grimaldi [4] defined a graph $G(\mathbb{Z}_n)$ based on the elements and units of $\mathbb{Z}_n$. The vertices of $G(\mathbb{Z}_n)$ are the elements of $\mathbb{Z}_n$ and distinct vertices x and y are defined to be adjacent if and only if $x+y$ is a unit of $\mathbb{Z}_n$. For a positive integer m, it follows that $G(\mathbb{Z}_{2m})$ is a $\varphi(2m)$-regular graph, where φ is the Euler phi function. In case $m \geq 2$, $G(\mathbb{Z}_{2m})$ can be expressed as the union of $\varphi(2m)/2$ Hamiltonian cycles. The odd case is not very easy, but the structure is clear and the results are similar to the even case. We recall that a *cone* over a graph is obtained by taking the categorical product of the graph and a path with a loop at one end, and then identifying all the vertices whose second coordinate is the other end of the path. When p is an odd prime, $G(\mathbb{Z}_p)$ can be expressed as a cone over a complete partite graph with $(p-1)/2$ partitions of size two. This leads to an explicit formula for the chromatic polynomial of $G(\mathbb{Z}_p)$. The paper [4] also concludes with some properties of the graphs $G(\mathbb{Z}_{p^m})$, where p is a prime number and $m \geq 2$.

Recently, Ashrafi et al. [1] generalized $G(\mathbb{Z}_n)$ to $G(R)$, the unit graph of R, where R is an arbitrary ring and studied the properties of this graph. Later, more properties of the unit graph of a ring and its applications were given in [7, 8, 9]. Let us first define this notion.

Definition 2.1. Let R be a ring and $U(R)$ be the set of unit elements of R. The *unit graph* of R, denoted by $G(R)$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x+y \in U(R)$.

For a given ring R, if we omit the word "distinct" in the definition of the unit graph $G(R)$, we obtain the *closed unit graph* of R which is denoted by $\overline{G}(R)$. This graph may have loops and if $2 \notin U(R)$, then $\overline{G}(R) = G(R)$.

The graphs in Fig. 1 are the unit graphs of the rings indicated.

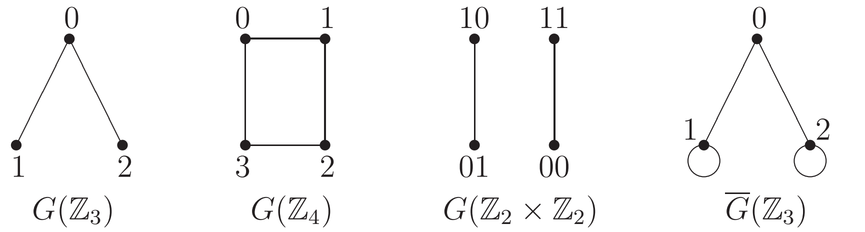

Figure 1. The unit graphs of some specific rings.

It is easy to see that, for given rings R and S, if $R \cong S$ as rings, then $G(R) \cong G(S)$ as graphs. This point is illustrated in Fig. 2, for the unit graphs of two isomorphic rings $\mathbb{Z}_3 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$.

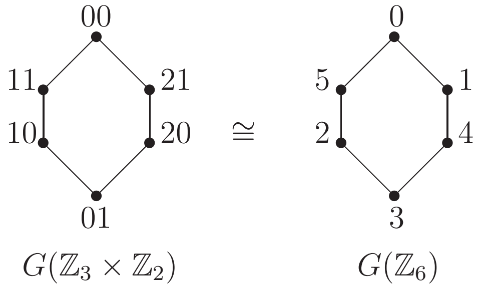

FIGURE 2. The unit graphs of two isomorphic rings.

For a graph G, let $V(G)$ denote the set of vertices of G. Let G_1 and G_2 be two vertex-disjoint graphs. The *categorical product* of G_1 and G_2 is denoted by $G_1 \dot{\times} G_2$. That is, $V(G_1 \dot{\times} G_2) := V(G_1) \times V(G_2)$ and two distinct vertices (x, y) and (x', y') are adjacent if and only if x is adjacent to x' in G_1 and y is adjacent to y' in G_2. Clearly, for given rings R_1 and R_2, two distinct vertices (x, y) and (x', y') of $\overline{G}(R_1) \dot{\times} \overline{G}(R_2)$ are adjacent if and only if x is adjacent to x' in $\overline{G}(R_1)$ and y is adjacent to y' in $\overline{G}(R_2)$. This implies that $\overline{G}(R_1) \dot{\times} \overline{G}(R_2) \cong G(R_1 \times R_2)$.

In Fig. 3, we illustrate the above point for the direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$.

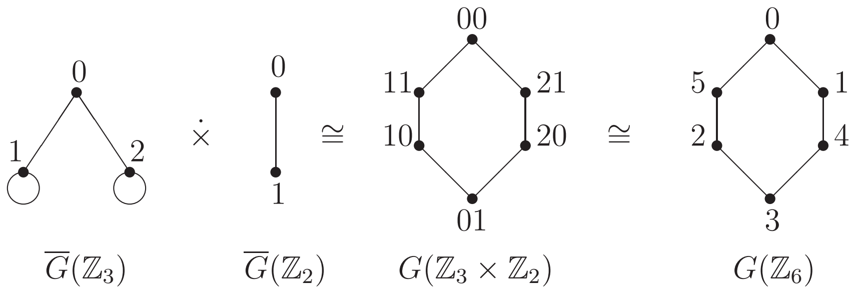

FIGURE 3. The categorical product of two closed unit graphs.

We are now in a position to state the main result of this paper.

Theorem 2.2. *Let R be a ring. Then the following statements hold true for the unit graph $G(R)$:*

(1) $\gamma(G(R)) = 1$ *if and only if R is a field.*

(2) $\gamma(G(R)) = 2$ *if and only if either R is a local ring which is not a field, R is isomorphic to the product of two fields such that only one of them has characteristic 2, or $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.*

(3) $\gamma(G(R)) = 3$ *if and only if* R *is not isomorphic to the product of two fields such that only one of them has characteristic* 2, *and* $R \cong R_1 \times R_2$, *where for* $i = 1, 2$, R_i *is a local ring with maximal ideal* $\mathfrak{m}_i$ *in such a way that* $R_i/\mathfrak{m}_i \ncong \mathbb{Z}_2$.

The rest of the paper is organized as follows. In Section 3, we deal with unit graphs associated with products of fields. By the results of this section, we give a proof for parts (1) and (2) of Theorem 2.2. In Section 4, we deal with unit graphs associated with products of local rings. The results of this section lead to a proof for part (3) of Theorem 2.2. In both sections, we state and prove some lemmas that will be used in the proof of Theorem 2.2. Furthermore, for the convenience of the reader, we state without proof a few known results in the form of propositions which will be used in the proofs. We also recall some definitions and notations concerning graphs for later use.

3. Unit graphs associated with products of fields

We start this section by unit graphs associated with fields. The following lemma gives us the domination number of such graphs.

Lemma 3.1. *Let* $\mathbb{F}$ *be a field. Then* $\gamma(G(\mathbb{F})) = 1$.

Proof. Since $\mathbb{F}$ is a field, every nonzero element of $\mathbb{F}$ is a unit element of $\mathbb{F}$. That is, for every $x \in \mathbb{F} \setminus \{0\}$, $x = x + 0 \in U(\mathbb{F})$ and so x is adjacent to 0. Hence, every nonzero element of $\mathbb{F}$ is adjacent to 0 and so $D = \{0\}$ is a dominating set of $G(\mathbb{F})$ with, of course, minimum cardinality. Therefore, we conclude that $\gamma(G(\mathbb{F})) = 1$. □

The degrees of all vertices of a unit graph is given by the following proposition. We recall that for a graph G and for a vertex x of G, the *degree* of x, denoted by $\deg(x)$, is the number of edges of G incident with x.

Proposition 3.2 ([1], Proposition 2.4)**.** *Let* R *be a ring. Then the following statements hold true for the unit graph* $G(R)$*:*

(1) *If* $2 \notin U(R)$, *then* $\deg(x) = |U(R)|$ *for every* $x \in R$.

(2) *If* $2 \in U(R)$, *then* $\deg(x) = |U(R)| - 1$ *for every* $x \in U(R)$ *and* $\deg(x) = |U(R)|$ *for every* $x \in R \setminus U(R)$.

We are now ready to prove part (1) of Theorem 2.2, that is:

Theorem 3.3. *Let* R *be a ring. Then* $\gamma(G(R)) = 1$ *if and only if* R *is a field.*

Proof. ($\Rightarrow$): Let $\gamma(G(R)) = 1$ and consider $D = \{x\}$ as a γ-set of $G(R)$. We claim that $\deg(x) = |U(R)|$. In order to prove the claim, we first suppose that $x = 0$. In this case, $x \notin U(R)$ and so by Proposition 3.2, we have $\deg(x) = |U(R)|$. Second, we suppose that $x \neq 0$. Note that we have $x = -x$; otherwise, since $D = \{x\}$ is a dominating set of $G(R)$, $-x$ is adjacent to x and so $-x + x = 0 \in U(R)$, which is a contradiction. Hence, $2x = 0$, which together with $x \neq 0$, imply that $2 \notin U(R)$. Now, by Proposition 3.2(1), we conclude that $\deg(x) = |U(R)|$. Therefore, the claim holds true. On the other hand, since $D = \{x\}$ is a dominating set of $G(R)$, every element of $R \setminus \{x\}$ is adjacent to x and so $\deg(x) = |R| - 1$. Therefore, $|U(R)| = |R| - 1$, which implies that R is a field.

($\Leftarrow$): This implication, by Lemma 3.1, is obvious. $\square$

We now state and prove some results that will be needed in the proof of part (2) of Theorem 2.2. In Lemmas 3.4 and 3.6, we give a lower bound for the domination number of two classes of unit graphs associated with products of two fields. Lemma 3.5 is useful for this purpose.

Lemma 3.4. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two fields in which both of them have characteristic 2 and none of them is isomorphic to $\mathbb{Z}_2$. Then $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \geq 3$.*

Proof. Suppose, in contrary, that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \leq 2$. Therefore, the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$ has a dominating set in the form of $D = \{(x_1, x_2), (x_3, x_4)\}$, where $x_1, x_3 \in \mathbb{F}_1$, $x_2, x_4 \in \mathbb{F}_2$ and $(x_1, x_2) \neq (x_3, x_4)$. Hence, we have either $x_1 \neq x_3$ or $x_2 \neq x_4$.

Case 1: Suppose that $x_1 \neq x_3$. The assumption $\mathrm{Char}(\mathbb{F}_1) = \mathrm{Char}(\mathbb{F}_2) = 2$ implies that

$$(x_1, x_4) + (x_1, x_2) = (0, x_4 + x_2) \notin U(\mathbb{F}_1 \times \mathbb{F}_2) \text{ and}$$

$$(x_1, x_4) + (x_3, x_4) = (x_1 + x_3, 0) \notin U(\mathbb{F}_1 \times \mathbb{F}_2).$$

Hence, (x_1, x_4) is not adjacent to the elements of D. Therefore, $(x_1, x_4) \in D$ and so $(x_1, x_4) = (x_1, x_2)$ or $(x_1, x_4) = (x_3, x_4)$. Now, $x_1 \neq x_3$ implies that the second equality does not hold and so the first one holds. Therefore, we have $x_4 = x_2$ and so $D = \{(x_1, x_2), (x_3, x_2)\}$. Let $a \in \mathbb{F}_1$ be given. Then $\mathrm{Char}(\mathbb{F}_2) = 2$ implies that

$$(a, x_2) + (x_1, x_2) = (a + x_1, 0) \notin U(\mathbb{F}_1 \times \mathbb{F}_2) \text{ and}$$

$$(a, x_2) + (x_3, x_2) = (a + x_3, 0) \notin U(\mathbb{F}_1 \times \mathbb{F}_2).$$

Hence, (a, x_2) is not adjacent to the elements of D. Therefore, $(a, x_2) \in D$ and so $(a, x_2) = (x_1, x_2)$ or $(a, x_2) = (x_3, x_2)$. This implies that $a = x_1$ or $a = x_3$. Since $a \in \mathbb{F}_1$ is arbitrary, we conclude that $\mathbb{F}_1 = \{x_1, x_3\} \cong \mathbb{Z}_2$, which is a contradiction.

Case 2: Suppose that $x_2 \neq x_4$. By a similar argument as in Case 1, this case leads to $\mathbb{F}_2 = \{x_2, x_4\} \cong \mathbb{Z}_2$, which is again a contradiction.

Therefore, we conclude that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \geq 3$, as required. $\square$

Lemma 3.5. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two fields. If $\mathrm{Char}(\mathbb{F}_1) \neq 2$ and $a \in \mathbb{F}_1$, then $D_1 = \{(a, 0), (-a, 0)\}$ is not a dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$. If $\mathrm{Char}(\mathbb{F}_2) \neq 2$ and $b \in \mathbb{F}_2$, then $D_2 = \{(0, b), (0, -b)\}$ is not a dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$.*

Proof. We prove the first part. The proof of the second part is similar. Suppose, in contrary, that $D_1 = \{(a, 0), (-a, 0)\}$ is a dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$.

Since $\mathrm{Char}(\mathbb{F}_1) \neq 2$, $|\mathbb{F}_1| \geq 3$ and so we may choose $x \in \mathbb{F}_1$ for which $x \neq a$ and $x \neq -a$. Now, we have

$$(x,0) + (a,0) = (x+a,0) \notin U(\mathbb{F}_1 \times \mathbb{F}_2) \text{ and}$$
$$(x,0) + (-a,0) = (x-a,0) \notin U(\mathbb{F}_1 \times \mathbb{F}_2).$$

Hence, $(x,0)$ is not adjacent to the elements of D_1. Therefore, $(x,0) \in D_1$ and so $(x,0) = (a,0)$ or $(x,0) = (-a,0)$. Hence, $x = a$ or $x = -a$, which contradicts $x \neq a$ and $x \neq -a$. Therefore, D_1 is not a dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$. □

Lemma 3.6. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two fields in which none of them has characteristic 2. Then $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \geq 3$.*

Proof. Suppose, in contrary, that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \leq 2$. Therefore, the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$ has a dominating set in the form of $D = \{(x_1,x_2),(x_3,x_4)\}$, where $x_1, x_3 \in \mathbb{F}_1$, $x_2, x_4 \in \mathbb{F}_2$ and $(x_1,x_2) \neq (x_3,x_4)$. Note that

$$(-x_1,-x_4) + (x_1,x_2) = (0,-x_4+x_2) \notin U(\mathbb{F}_1 \times \mathbb{F}_2) \text{ and}$$
$$(-x_1,-x_4) + (x_3,x_4) = (-x_1+x_3,0) \notin U(\mathbb{F}_1 \times \mathbb{F}_2).$$

Hence, $(-x_1,-x_4)$ is not adjacent to the elements of D. Therefore, $(-x_1,-x_4) \in D$ and so $(-x_1,-x_4) = (x_1,x_2)$ or $(-x_1,-x_4) = (x_3,x_4)$. Also,

$$(-x_3,-x_2) + (x_1,x_2) = (-x_3+x_1,0) \notin U(\mathbb{F}_1 \times \mathbb{F}_2) \text{ and}$$
$$(-x_3,-x_2) + (x_3,x_4) = (0,-x_2+x_4) \notin U(\mathbb{F}_1 \times \mathbb{F}_2).$$

Hence, $(-x_3,-x_2)$ is not adjacent to the elements of D. Therefore, $(-x_3,-x_2) \in D$ and so $(-x_3,-x_2) = (x_1,x_2)$ or $(-x_3,-x_2) = (x_3,x_4)$. Hence, one of the following four possibilities occurs:

(1) $(-x_1,-x_4) = (x_1,x_2)$ and $(-x_3,-x_2) = (x_1,x_2)$,
(2) $(-x_1,-x_4) = (x_3,x_4)$ and $(-x_3,-x_2) = (x_3,x_4)$,
(3) $(-x_1,-x_4) = (x_3,x_4)$ and $(-x_3,-x_2) = (x_1,x_2)$, or
(4) $(-x_1,-x_4) = (x_1,x_2)$ and $(-x_3,-x_2) = (x_3,x_4)$.

The first two cases lead to $x_1 = x_2 = x_3 = x_4 = 0$, which contradicts $(x_1,x_2) \neq (x_3,x_4)$. The third case leads to $x_1 = -x_3 := a$ and $x_2 = x_4 = 0$. Therefore, $D = \{(a,0),(-a,0)\}$, which contradicts Lemma 3.5. The fourth case leads to $x_1 = x_3 = 0$ and $x_2 = -x_4 := b$. Therefore, $D = \{(0,b),(0,-b)\}$, which again contradicts Lemma 3.5. Therefore, we conclude that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \geq 3$, as required. □

In Lemma 3.12, we prove that the domination number of unit graphs associated with products of three fields is equal to four. The following results are useful for this purpose.

We recall that a *bipartite* graph is one whose vertex set is partitioned into two (not necessarily nonempty) disjoint subsets, called *partite sets*, in such a way that the two end vertices for each edge lie in distinct partite sets.

Proposition 3.7 ([1], Theorem 3.5). *Let R be a ring and $\mathfrak{m}$ be a maximal ideal of R such that $|R/\mathfrak{m}| = 2$. Then the unit graph $G(R)$ is a bipartite graph with partite sets $V_1 = \mathfrak{m}$ and $V_2 = R \setminus \mathfrak{m}$.*

Lemma 3.8. *Let $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$ be three fields such that one is isomorphic to $\mathbb{Z}_2$, one is not isomorphic to $\mathbb{Z}_2$ and the latter one has characteristic 2. Then $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$.*

Proof. Without loss of generality, we may assume that $\mathbb{F}_1 \ncong \mathbb{Z}_2$, $\mathrm{Char}(\mathbb{F}_2) = 2$ and $\mathbb{F}_3 \cong \mathbb{Z}_2$. Since $\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$ is a ring with maximal ideal $\mathbb{F}_1 \times \mathbb{F}_2 \times \{0\}$ such that $|(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)/(\mathbb{F}_1 \times \mathbb{F}_2 \times \{0\})| = 2$, Proposition 3.7 implies that the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)$ is bipartite with partite sets $V_1 = \mathbb{F}_1 \times \mathbb{F}_2 \times \{0\}$ and $V_2 = \mathbb{F}_1 \times \mathbb{F}_2 \times \{1\}$. This implies that $|V_1| = |V_2| = |\mathbb{F}_1 \times \mathbb{F}_2|$.

Assume now, in contrary, that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \leq 3$ and consider D with $|D| = 3$ as a dominating set of $G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)$. Note that $|\mathbb{F}_1| \geq 3$, therefore $|\mathbb{F}_1 \times \mathbb{F}_2| \geq 6$ and so $|V_1| = |V_2| \geq 6$. Now, $|D| = 3$ implies that D cannot completely lie either in V_1 or V_2. Hence, $D \cap V_1 \neq \emptyset$ and $D \cap V_2 \neq \emptyset$. This implies that either $|D \cap V_1| = 1$ or $|D \cap V_2| = 1$. Without loss of generality, suppose $|D \cap V_1| = 1$ is the case. Since $D \cap V_1 \subseteq V_1 = \mathbb{F}_1 \times \mathbb{F}_2 \times \{0\}$, we may consider $D \cap V_1 = \{(x, y, 0)\}$ with $x \in \mathbb{F}_1$ and $y \in \mathbb{F}_2$. Since $|\mathbb{F}_1| \geq 3$, we may choose three distinct elements x_1, x_2 and x_3 of $\mathbb{F}_1$. Therefore, $(x_1, y, 1)$, $(x_2, y, 1)$ and $(x_3, y, 1)$ are three distinct elements of V_2. But $\mathrm{Char}(\mathbb{F}_2) = 2$ implies that for $i = 1, 2, 3$,

$$(x_i, y, 1) + (x, y, 0) = (x_i + x, 0, 1) \notin U(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)$$

and so $(x_i, y, 1)$ is not adjacent to $(x, y, 0)$. Hence, for $i = 1, 2, 3$, $(x_i, y, 1) \in D$ and so $|D| \geq 4$, which is a contradiction. This shows that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$. □

We need the following result from [11] for later use. Note that in [11], the author has used a different notion of product. But it causes no problem here as loops have no influence on the domination number. Here, K_1 denotes the graph with one vertex and no edges.

Proposition 3.9 ([11], Theorem 3.1 and Proposition 4.1). *Let G and H be two graphs. Then $\gamma(G \dot{\times} H) \geq \gamma(G) + \gamma(H) - 1$. Moreover, if the equality holds and $\gamma(G) = 1$, then $G = K_1$ and H is an edgeless graph.*

Lemma 3.10. *Let R be a ring and $\mathbb{F}$ be a field. Then $\gamma(G(R \times \mathbb{F})) > \gamma(G(R))$.*

Proof. By using Proposition 3.9, we conclude that

$$\gamma\big(\overline{G}(R) \dot{\times} \overline{G}(\mathbb{F})\big) \geq \gamma\big(\overline{G}(R)\big) + \gamma\big(\overline{G}(\mathbb{F})\big) - 1.$$

Note that, again by Proposition 3.9, if the equality holds, then since, by Lemma 3.1, $\gamma\big(\overline{G}(\mathbb{F})\big) = \gamma(G(\mathbb{F})) = 1$, we obtain that $\overline{G}(\mathbb{F}) = K_1$, which is a contradiction. Therefore, the equality does not hold, that is,

$$\gamma\big(\overline{G}(R) \dot{\times} \overline{G}(\mathbb{F})\big) > \gamma\big(\overline{G}(R)\big) + \gamma\big(\overline{G}(\mathbb{F})\big) - 1 = \gamma(\overline{G}(R)).$$

Now, since for every ring S, $\gamma\big(\overline{G}(S)\big) = \gamma(G(S))$, we conclude that

$$\gamma(G(R \times \mathbb{F})) = \gamma\big(\overline{G}(R)\dot{\times}\overline{G}(\mathbb{F})\big) > \gamma\big(\overline{G}(R)\big) = \gamma(G(R)),$$

as required. $\square$

Lemma 3.11. *Let $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$ be three fields. If $a \in \mathbb{F}_1$, $b \in \mathbb{F}_2$ and $c \in \mathbb{F}_3$ are nonzero elements, then $D = \{(0,0,0), (-a,-b,0), (a,0,-c), (0,b,c)\}$ is a dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)$.*

Proof. A given element in $(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3) \setminus D$ has one of the following forms, where $x \in \mathbb{F}_1$, $y \in \mathbb{F}_2$ and $z \in \mathbb{F}_3$ are nonzero elements: either $(x,0,0)$, $(0,y,0)$, $(0,0,z)$, $(x,y,0)$ with $x \neq -a$ or $y \neq -b$, $(x,0,z)$ with $x \neq a$ or $z \neq -c$, $(0,y,z)$ with $y \neq b$ or $z \neq c$, or (x,y,z).

For $(x,0,0)$, we have

$$(x,0,0) + (0,b,c) = (x,b,c) \in U(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)$$

and so $(x,0,0)$ is adjacent to $(0,b,c)$. By a similar argument, we obtain that $(0,y,0)$ is adjacent to $(a,0,-c)$ and $(0,0,z)$ is adjacent to $(-a,-b,0)$.

For $(x,y,0)$ with $x \neq -a$ or $y \neq -b$, we have either

$$(x,y,0) + (a,0,-c) = (x+a,y,-c) \in U(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3) \text{ or}$$
$$(x,y,0) + (0,b,c) = (x,y+b,c) \in U(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3).$$

This implies that $(x,y,0)$ with $x \neq -a$ or $y \neq -b$ is adjacent to $(a,0,-c)$ or $(0,b,c)$. By a similar argument, we obtain that $(x,0,z)$ with $x \neq a$ or $z \neq -c$ is adjacent to $(-a,-b,0)$ or $(0,b,c)$. Also, $(0,y,z)$ with $y \neq b$ or $z \neq c$ is adjacent to $(-a,-b,0)$ or $(a,0,-c)$.

Finally, for (x,y,z), we have

$$(x,y,z) + (0,0,0) = (x,y,z) \in U(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)$$

and so (x,y,z) is adjacent to $(0,0,0)$.

Therefore, D is a dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)$. $\square$

Lemma 3.12. *Let $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$ be three fields. Then $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) = 4$.*

Proof. We first show that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$. In order to do this, note that there are two possibilities: either at least two of $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$, say $\mathbb{F}_1$ and $\mathbb{F}_2$, have characteristic 2, or at least two of $\mathbb{F}_1$, $\mathbb{F}_2$ and $\mathbb{F}_3$, say again $\mathbb{F}_1$ and $\mathbb{F}_2$, do not have characteristic 2.

First, assume that $\mathrm{Char}(\mathbb{F}_1) = 2$ and $\mathrm{Char}(\mathbb{F}_2) = 2$. Now, one of the following cases may occur:

Case 1: Both of $\mathbb{F}_1$ and $\mathbb{F}_2$ are isomorphic to $\mathbb{Z}_2$. In this case, if $\mathbb{F}_3 \cong \mathbb{Z}_2$, then it is easy to see that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) = 4$. If $\mathbb{F}_3 \ncong \mathbb{Z}_2$, then we have $\mathbb{F}_1 \cong \mathbb{Z}_2$, $\mathrm{Char}(\mathbb{F}_2) = 2$ and $\mathbb{F}_3 \ncong \mathbb{Z}_2$, and so by Lemma 3.8, $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$.

Case 2: One of $\mathbb{F}_1$ or $\mathbb{F}_2$ is isomorphic to $\mathbb{Z}_2$ and the other one is not, say $\mathbb{F}_1 \cong \mathbb{Z}_2$ and $\mathbb{F}_2 \ncong \mathbb{Z}_2$. In this case, if $\mathbb{F}_3 \cong \mathbb{Z}_2$, then we have $\mathrm{Char}(\mathbb{F}_1) = 2$, $\mathbb{F}_2 \ncong \mathbb{Z}_2$ and $\mathbb{F}_3 \cong \mathbb{Z}_2$, and so by Lemma 3.8, $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$. If $\mathbb{F}_3 \ncong \mathbb{Z}_2$, then we have $\mathbb{F}_1 \cong \mathbb{Z}_2$, $\mathrm{Char}(\mathbb{F}_2) = 2$ and $\mathbb{F}_3 \ncong \mathbb{Z}_2$, and so again by Lemma 3.8, $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$.

Case 3: None of $\mathbb{F}_1$ and $\mathbb{F}_2$ is isomorphic to $\mathbb{Z}_2$. In this case, by using Lemmas 3.10 and 3.4, we obtain that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) > \gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \geq 3$ and so $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$.

Second, assume that $\mathrm{Char}(\mathbb{F}_1) \neq 2$ and $\mathrm{Char}(\mathbb{F}_2) \neq 2$. In this case, by using Lemmas 3.10 and 3.6, we obtain that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) > \gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \geq 3$ and so $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$.

All in all, we have proven that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) \geq 4$ and in view of Lemma 3.11, we conclude that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3)) = 4$, as required. $\square$

In Lemma 3.16, we give a lower bound for the domination number of unit graphs associated with rings which are products of more than two local rings. The following results are useful for this purpose.

Lemma 3.13. *Let $\mathbb{F}_1, \ldots, \mathbb{F}_n$ be fields. If $n \geq 3$, then $\gamma(G(\mathbb{F}_1 \times \cdots \times \mathbb{F}_n)) \geq n + 1$.*

Proof. We prove the lemma by induction on n. By Lemma 3.12, the inequality is true for $n = 3$ and we assume that it is true for $n - 1$. Now, by using Lemma 3.10 and the induction hypothesis, we obtain that $\gamma(G(\mathbb{F}_1 \times \cdots \times \mathbb{F}_n)) > \gamma(G(\mathbb{F}_1 \times \cdots \times \mathbb{F}_{n-1})) \geq n$ and so $\gamma(G(\mathbb{F}_1 \times \cdots \times \mathbb{F}_n)) \geq n + 1$. $\square$

Lemma 3.14. *Let R be a ring. Then $\gamma(G(R/J(R))) \leq \gamma(G(R))$, where $J(R)$ denotes the Jacobson radical of R.*

Proof. Let $\gamma(G(R)) = n$ and consider $D = \{x_1, \ldots, x_n\}$ as a γ-set of $G(R)$. Among $x_1 + J(R), \ldots, x_n + J(R)$, choose distinct ones and call them $x_{i_1} + J(R), \ldots, x_{i_k} + J(R)$. Note that $\{x_{i_1}, \ldots, x_{i_k}\} \subseteq \{x_1, \ldots, x_n\}$ and, in particular, $k \leq n$. We now claim that $D' = \{x_{i_1} + J(R), \ldots, x_{i_k} + J(R)\}$ is a dominating set of the unit graph $G(R/J(R))$. In order to prove the claim, let $y + J(R) \in (R/J(R)) \setminus D'$ be given. Therefore, $y \in R \setminus D$ and so there exists $x_\ell \in D$ such that y is adjacent to x_ℓ in $G(R)$, that is, $y + x_\ell \in U(R)$. This implies that $(y + J(R)) + (x_\ell + J(R)) \in U(R/J(R))$ and so $y + J(R)$ is adjacent to $x_\ell + J(R)$ in $G(R/J(R))$. But, $x_\ell + J(R) = x_{i_j} + J(R)$ for some $1 \leq j \leq k$. Hence, $y + J(R)$ is adjacent to $x_{i_j} + J(R)$ in $G(R/J(R))$ and so the claim holds true. Therefore, we have $\gamma(G(R/J(R))) \leq |D'| = k \leq n = \gamma(G(R))$. $\square$

We note that Lemma 3.14 remains valid for every arbitrary proper ideal of R, but in this paper we shall apply it in the above form.

Remark 3.15. It is known that every finite commutative ring is isomorphic to a direct product of local rings (see [10, Page 95]). Thus for a given ring R, we may write $R \cong R_1 \times \cdots \times R_n$, where for every $1 \leq i \leq n$, R_i is a local ring with maximal ideal $\mathfrak{m}_i$. It is easy to see that

$$\{R_1 \times \cdots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \cdots \times R_n \mid 1 \leq i \leq n\}$$

is the set of all maximal ideals of $R_1 \times \cdots \times R_n$ and thus we conclude that the Jacobson radical of R is isomorphic to $J(R_1 \times \cdots \times R_n) = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_n$. Also, note that the function $f : R_1 \times \cdots \times R_n \longrightarrow (R_1/\mathfrak{m}_1) \times \cdots \times (R_n/\mathfrak{m}_n)$, given by $f(r_1, \ldots, r_n) = (r_1 + \mathfrak{m}_1, \ldots, r_n + \mathfrak{m}_n)$, is an epimorphism with $\ker f = \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_n$ and thus we have

$$\frac{R}{J(R)} \cong \frac{R_1 \times \cdots \times R_n}{\mathfrak{m}_1 \times \cdots \times \mathfrak{m}_n} \cong \frac{R_1}{\mathfrak{m}_1} \times \cdots \times \frac{R_n}{\mathfrak{m}_n}.$$

Lemma 3.16. *Let R be a ring and in view of Remark* 3.15*, write $R \cong R_1 \times \cdots \times R_n$, where for every $1 \leq i \leq n$, R_i is a local ring. If $n \geq 3$, then $\gamma(G(R)) \geq n + 1$.*

Proof. For every $1 \leq i \leq n$, let $\mathfrak{m}_i$ be the maximal ideal of R_i. By using Remark 3.15, we conclude that $R/J(R) \cong \mathbb{F}_1 \times \cdots \times \mathbb{F}_n$, where for every $1 \leq i \leq n$, $\mathbb{F}_i$ is the field $R_i/\mathfrak{m}_i$. Now, by using Lemmas 3.14 and 3.13, we conclude that $\gamma(G(R)) \geq \gamma(G(R/J(R))) = \gamma(G(\mathbb{F}_1 \times \cdots \times \mathbb{F}_n)) \geq n + 1$, as required. $\square$

The following three lemmas are needed for the proof of part (2) of Theorem 2.2.

Lemma 3.17. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and let $x \in R$. If $x \notin \mathfrak{m}$, then $x \in U(R)$. If $x \in \mathfrak{m}$, then $x + 1 \in U(R)$. In particular, $D = \{0, 1\}$ is a dominating set of the unit graph $G(R)$.*

Proof. First, let $x \notin \mathfrak{m}$. If $x \notin U(R)$, then $\langle x \rangle$ is a proper ideal of R and since R is local, $\langle x \rangle \subseteq \mathfrak{m}$. Hence, $x \in \mathfrak{m}$, which contradicts $x \notin \mathfrak{m}$. Therefore, $x \in U(R)$.

Second, let $x \in \mathfrak{m}$. Then $x + 1 \notin \mathfrak{m}$; otherwise, we obtain that $1 \in \mathfrak{m}$, which is a contradiction. Now, by the previous case, $x + 1 \in U(R)$.

In particular, for a given $a \in R$, if $a \notin \mathfrak{m}$, then it is adjacent to 0 and if $a \in \mathfrak{m}$, then it is adjacent to 1. Therefore, $D = \{0, 1\}$ is a dominating set of $G(R)$. $\square$

We recall that for a graph G and for a given vertex $x \in V(G)$, the *neighbor set* of x is the set $N(x) := \{v \in V(G) \mid v \text{ is adjacent to } x\}$.

Lemma 3.18. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two fields with* $\mathrm{Char}(\mathbb{F}_1) = 2$ *and* $\mathrm{Char}(\mathbb{F}_2) \neq 2$*. If $b \in \mathbb{F}_2$ is nonzero, then $D = \{(0, -b), (1, b)\}$ is a dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$.*

Proof. Since $\mathrm{Char}(\mathbb{F}_1) = 2$, we may conclude that $2(1, 1) = (0, 2) \notin U(\mathbb{F}_1 \times \mathbb{F}_2)$. Therefore, Proposition 3.2(1) implies that the degree of every element of $G(\mathbb{F}_1 \times \mathbb{F}_2)$ is equal to $|U(\mathbb{F}_1 \times \mathbb{F}_2)| = |U(\mathbb{F}_1)||U(\mathbb{F}_2)| = (|\mathbb{F}_1| - 1)(|\mathbb{F}_2| - 1)$. In particular, we conclude that

$$|N(0, -b)| = |N(1, b)| = (|\mathbb{F}_1| - 1)(|\mathbb{F}_2| - 1).$$

Also, for a given element $(x, y) \in \mathbb{F}_1 \times \mathbb{F}_2$, it is adjacent to both of $(0, -b)$ and $(1, b)$ if and only if

$$(x, y) + (0, -b) = (x, y - b) \in U(\mathbb{F}_1 \times \mathbb{F}_2) \text{ and}$$
$$(x, y) + (1, b) = (x + 1, y + b) \in U(\mathbb{F}_1 \times \mathbb{F}_2),$$

if and only if $x \neq 0, -1$ and $y \neq b, -b$. Therefore,

$$N(0, -b) \cap N(1, b) = \{(x, y) \in \mathbb{F}_1 \times \mathbb{F}_2 \mid x \neq 0, -1 \text{ and } y \neq b, -b\}.$$

Since $\mathrm{Char}(\mathbb{F}_2) \neq 2$ and $b \in \mathbb{F}_2$ is nonzero, we obtain that

$$|N(0,-b) \cap N(1,b)| = (|\mathbb{F}_1| - 2)(|\mathbb{F}_2| - 2).$$

Now, the inclusion-exclusion principle implies that

$$\begin{aligned} |N(0,-b) \cup N(1,b)| &= |N(0,-b)| + |N(1,b)| - |N(0,-b) \cap N(1,b)| \\ &= 2(|\mathbb{F}_1| - 1)(|\mathbb{F}_2| - 1) - (|\mathbb{F}_1| - 2)(|\mathbb{F}_2| - 2) \\ &= |\mathbb{F}_1||\mathbb{F}_2| - 2 \\ &= |\mathbb{F}_1 \times \mathbb{F}_2| - 2. \end{aligned}$$

Since

$$(0,-b) + (1,b) = (1,0) \notin U(\mathbb{F}_1 \times \mathbb{F}_2),$$

$(0,-b)$ and $(1,b)$ are not adjacent and so $(0,-b), (1,b) \notin N(0,-b) \cup N(1,b)$. This implies that $N(0,-b) \cup N(1,b) = (\mathbb{F}_1 \times \mathbb{F}_2) \setminus D$. Therefore, every element of $(\mathbb{F}_1 \times \mathbb{F}_2) \setminus D$ is adjacent to at least an element of D, which implies that D is a dominating set of $G(\mathbb{F}_1 \times \mathbb{F}_2)$. □

Lemma 3.19. *Let $\mathbb{F}$ be a field. Then $D = \{(0,0), (1,0)\}$ is a dominating set of the unit graph $G(\mathbb{Z}_2 \times \mathbb{F})$.*

Proof. A given element in $(\mathbb{Z}_2 \times \mathbb{F}) \setminus D$ has one of the following forms, where $y_1, y_2 \in \mathbb{F}$ are nonzero elements: either $(0, y_1)$ or $(1, y_2)$. Since

$$(0, y_1) + (1, 0) = (1, y_1) \in U(\mathbb{Z}_2 \times \mathbb{F}) \text{ and}$$

$$(1, y_2) + (0, 0) = (1, y_2) \in U(\mathbb{Z}_2 \times \mathbb{F}),$$

$(0, y_1)$ is adjacent to $(1, 0)$ and $(1, y_2)$ is adjacent to $(0, 0)$. Therefore, D is a dominating set of $G(\mathbb{Z}_2 \times \mathbb{F})$. □

We are now ready to prove part (2) of Theorem 2.2, that is:

Theorem 3.20. *Let R be a ring. Then $\gamma(G(R)) = 2$ if and only if either R is a local ring which is not a field, R is isomorphic to the product of two fields such that only one of them has characteristic 2, or $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.*

Proof. ($\Rightarrow$): Let $\gamma(G(R)) = 2$. By Remark 3.15, we may write $R \cong R_1 \times \cdots \times R_n$, where for every $1 \leq i \leq n$, R_i is a local ring. If $n \geq 3$, then by using Lemma 3.16, we conclude that $2 = \gamma(G(R)) \geq n + 1 \geq 4$, which is a contradiction. Therefore, $n \leq 2$ and so either $R \cong R_1$ or $R \cong R_1 \times R_2$. In the first case, R is a local ring which, by Lemma 3.1, is not a field. In the second case, by using Proposition 3.9 and the fact that for every ring S, $\gamma\big(\overline{G}(S)\big) = \gamma(G(S))$, we may conclude that

$$\begin{aligned} 2 &= \gamma(G(R)) \\ &= \gamma(G(R_1 \times R_2)) \\ &= \gamma\big(\overline{G}(R_1)\dot{\times}\overline{G}(R_2)\big) \\ &\geq \gamma\big(\overline{G}(R_1)\big) + \gamma\big(\overline{G}(R_2)\big) - 1 \\ &= \gamma(G(R_1)) + \gamma(G(R_2)) - 1. \end{aligned}$$

Since for $i = 1, 2$, we have $\gamma(G(R_i)) \geq 1$, we obtain that

$$2 \leq \gamma(G(R_1)) + \gamma(G(R_2)) \leq 3,$$

which implies that either $\gamma(G(R_1)) + \gamma(G(R_2)) = 2$ or $\gamma(G(R_1)) + \gamma(G(R_2)) = 3$. If $\gamma(G(R_1)) + \gamma(G(R_2)) = 3$ is the case, then one of its terms is equal to one and the other one is equal to two. For example, say $\gamma(G(R_1)) = 2$ and $\gamma(G(R_2)) = 1$. Therefore, by Theorem 3.3, we conclude that $R_2 := \mathbb{F}_2$ is a field and so $R \cong R_1 \times \mathbb{F}_2$. Now, by Lemma 3.10, $2 = \gamma(G(R)) = \gamma(G(R_1 \times \mathbb{F}_2)) > \gamma(G(R_1)) = 2$, which is a contradiction and so this case does not hold. Therefore, we have $\gamma(G(R_1)) + \gamma(G(R_2)) = 2$. This implies that $\gamma(G(R_1)) = \gamma(G(R_2)) = 1$. Therefore, by Theorem 3.3, we conclude that $R_1 := \mathbb{F}_1$ and $R_2 := \mathbb{F}_2$ are fields and also $R \cong \mathbb{F}_1 \times \mathbb{F}_2$. Now, in view of Lemmas 3.4 and 3.6, either R is isomorphic to the product of two fields such that only one of them has characteristic 2, or $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field.

($\Leftarrow$): Let R be one of the rings in the statement of the theorem. It is easy to see that R is not a field and so by Theorem 3.3, we conclude that $\gamma(G(R)) \neq 1$. Now, Lemmas 3.17, 3.18 and 3.19 imply that R has a dominating set with two elements and so $\gamma(G(R)) = 2$. □

4. Unit graphs associated with products of local rings

We now state and prove some results that will be needed in the proof of part (3) of Theorem 2.2.

We start by introducing the notion of a total dominating set. Let G be a graph with vertex set V. A subset D of V is called a *total dominating set* of G if every vertex in V is adjacent to at least a vertex in D. In other words, D dominates not only vertices outside D but also vertices in D. A *γ_t-set* of G is a minimum total dominating set of G, that is, a total dominating set of G whose cardinality is minimum. The *total domination number* of G, denoted by $\gamma_t(G)$, is the cardinality of a γ_t-set of G. It is obvious that $\gamma(G) \leq \gamma_t(G)$.

In Lemma 4.4, we give the domination number and the total domination number of a class of unit graphs associated with products of two local rings. The following lemmas are useful for this purpose.

Lemma 4.1. *Let R be a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$ and let S be a local ring. Then $\gamma_t(G(R \times S)) \leq 4$.*

Proof. Since $R \times S$ is a ring with maximal ideal $\mathfrak{m} \times S$ such that $|(R \times S)/(\mathfrak{m} \times S)| = 2$, Proposition 3.7 implies that the unit graph $G(R \times S)$ is bipartite with partite sets $V_1 = \mathfrak{m} \times S$ and $V_2 = (R \setminus \mathfrak{m}) \times S$.

We now claim that $D = \{(0,0),(0,1),(1,0),(1,1)\}$ is a total dominating set of $G(R \times S)$. In order to prove the claim, note that for every $(a,b) \in R \times S$, we have either $(a,b) \in V_1 = \mathfrak{m} \times S$ or $(a,b) \in V_2 = (R \setminus \mathfrak{m}) \times S$. We now apply Lemma 3.17 to get the following observations. Let $\mathfrak{n}$ be the maximal ideal of S. In the first case, that is, $(a,b) \in \mathfrak{m} \times S$, if $b \notin \mathfrak{n}$, then

$$(a,b) + (1,0) = (a+1,b) \in U(R \times S)$$

and so (a,b) is adjacent to $(1,0)$; and if $b \in \mathfrak{n}$, then

$$(a,b) + (1,1) = (a+1,b+1) \in U(R \times S)$$

and so (a,b) is adjacent to $(1,1)$. In the second case, that is, $(a,b) \in (R \setminus \mathfrak{m}) \times S$, if $b \notin \mathfrak{n}$, then

$$(a,b) + (0,0) = (a,b) \in U(R \times S)$$

and so (a,b) is adjacent to $(0,0)$; and if $b \in \mathfrak{n}$, then

$$(a,b) + (0,1) = (a,b+1) \in U(R \times S)$$

and so (a,b) is adjacent to $(0,1)$. Therefore, the claim holds true and so $\gamma_t(G(R \times S)) \leq 4$, as required. $\square$

Lemma 4.2. *Let R be a local ring with maximal ideal $\mathfrak{m}$ such that $|R/\mathfrak{m}| = 2$ and let S be a local ring. If R is not a field, then $\gamma(G(R \times S)) = \gamma_t(G(R \times S)) = 4$.*

Proof. Since $R \times S$ is a ring with maximal ideal $\mathfrak{m} \times S$ such that $|(R \times S)/(\mathfrak{m} \times S)| = 2$, Proposition 3.7 implies that the unit graph $G(R \times S)$ is bipartite with partite sets $V_1 = \mathfrak{m} \times S$ and $V_2 = (R \setminus \mathfrak{m}) \times S$. Note that $|R/\mathfrak{m}| = 2$ implies that $|\mathfrak{m}| = |R \setminus \mathfrak{m}| = |R|/2$ and so $|V_1| = |V_2| = (|R|/2)|S|$.

Assume now, in contrary, that $\gamma(G(R \times S)) \leq 3$ and consider D with $|D| = 3$ as a dominating set of $G(R \times S)$. Since R is not a field, $|R| \geq 4$. Also, S is a ring and so $|S| \geq 2$. Therefore, $|V_1| = |V_2| = (|R|/2)|S| \geq 4$. Now, $|D| = 3$ implies that D cannot completely lie either in V_1 or V_2. Hence, $D \cap V_1 \neq \emptyset$ and $D \cap V_2 \neq \emptyset$. This implies that either $|D \cap V_1| = 1$ or $|D \cap V_2| = 1$. Without loss of generality, suppose $|D \cap V_1| = 1$ is the case. Since $D \cap V_1 \subseteq V_1 = \mathfrak{m} \times S$, we may consider $D \cap V_1 = \{(x,y)\}$ with $x \in \mathfrak{m}$ and $y \in S$. Since $|R \setminus \mathfrak{m}| = |R|/2 \geq 2$, we may choose distinct elements x_1 and x_2 of $R \setminus \mathfrak{m}$. Therefore, $(x_1,-y)$ and $(x_2,-y)$ are distinct elements of $(R \setminus \mathfrak{m}) \times S = V_2$. Also, $(x,y) \in V_1$ implies that $(x,y) \neq (x_1,-y)$ and $(x,y) \neq (x_2,-y)$. Note that, if $(x_1,-y) \notin D$, then $(x_1,-y)$ is adjacent to (x,y) and so

$$(x_1,-y) + (x,y) = (x_1+x,0) \in U(R \times S),$$

which is impossible. Therefore, $(x_1, -y) \in D$. A similar argument shows that $(x_2, -y) \in D$. Hence, $D = \{(x, y), (x_1, -y), (x_2, -y)\}$. Since $|\mathfrak{m}| = |R|/2 \geq 2$, we may choose $a \in \mathfrak{m} \setminus \{x\}$. Now, $(a, y) \in \mathfrak{m} \times S = V_1$ and $(a, y) \neq (x, y)$ imply that $(a, y) \notin D$. Since $(a, y) \in V_1$, it cannot be adjacent to (x, y) and so it is either adjacent to $(x_1, -y)$ or $(x_2, -y)$. Therefore, we have either

$$(a, y) + (x_1, -y) = (a + x_1, 0) \in U(R \times S) \text{ or}$$

$$(a, y) + (x_2, -y) = (a + x_2, 0) \in U(R \times S),$$

which both of them are contradictions. This shows that $\gamma(G(R \times S)) \geq 4$.

Now, by Lemma 4.1, we have $4 \leq \gamma(G(R \times S)) \leq \gamma_t(G(R \times S)) \leq 4$ and so $\gamma(G(R \times S)) = \gamma_t(G(R \times S)) = 4$, as required. □

Lemma 4.3. *Let S be a local ring which is not a field. Then $\gamma(G(\mathbb{Z}_2 \times S)) = \gamma_t(G(\mathbb{Z}_2 \times S)) = 4$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of S. There are two possibilities: either $|S/\mathfrak{m}| = 2$ or $|S/\mathfrak{m}| \geq 3$. If $|S/\mathfrak{m}| = 2$ is the case, then by Lemma 4.2, we conclude that $\gamma(G(\mathbb{Z}_2 \times S)) = \gamma_t(G(\mathbb{Z}_2 \times S)) = 4$, as required. Therefore, we suppose that $|S/\mathfrak{m}| \geq 3$. Since $\mathbb{Z}_2 \times S$ is a ring with maximal ideal $\{0\} \times S$ such that $|(\mathbb{Z}_2 \times S)/(\{0\} \times S)| = 2$, Proposition 3.7 implies that the unit graph $G(\mathbb{Z}_2 \times S)$ is bipartite with partite sets $V_1 = \{0\} \times S$ and $V_2 = \{1\} \times S$. This implies that $|V_1| = |V_2| = |S|$.

We now claim that $\gamma(G(\mathbb{Z}_2 \times S)) \geq 4$. In order to prove the claim, in contrary, suppose that $\gamma(G(\mathbb{Z}_2 \times S)) \leq 3$ and consider D with $|D| = 3$ as a dominating set of $G(\mathbb{Z}_2 \times S)$. Since S is not a field, $|\mathfrak{m}| \geq 2$. Therefore, $|S/\mathfrak{m}| \geq 3$ implies that $|S| \geq 3|\mathfrak{m}| \geq 6$ and so $|V_1| = |V_2| = |S| \geq 6$. Now, $|D| = 3$ implies that D cannot completely lie either in V_1 or V_2. Hence, $D \cap V_1 \neq \emptyset$ and $D \cap V_2 \neq \emptyset$. This implies that either $|D \cap V_1| = 1$ or $|D \cap V_2| = 1$. Without loss of generality, suppose $|D \cap V_1| = 1$ is the case. Therefore, $|D \cap V_2| = 2$ and so we may consider $D = \{(0, y_1), (1, y_2), (1, y_3)\}$, where $y_1, y_2, y_3 \in S$ with $y_2 \neq y_3$. Let $y \in S \setminus \{y_2, y_3\}$ be given. Therefore, $(1, y) \notin D$ and so it is adjacent to an element of D. Since $(1, y)$, $(1, y_2)$ and $(1, y_3)$ are elements of V_2, $(1, y)$ is not adjacent to $(1, y_2)$ and $(1, y_3)$, therefore it is adjacent to $(0, y_1)$. Hence, for every $y \in S \setminus \{y_2, y_3\}$, $(1, y)$ is adjacent to $(0, y_1)$, which implies that $\deg(0, y_1) \geq |S| - 2$. On the other hand, by Proposition 3.2 and Lemma 3.17,

$$\deg(0, y_1) \leq |U(\mathbb{Z}_2 \times S)| = |U(\mathbb{Z}_2)||U(S)| = |U(S)| = |S| - |\mathfrak{m}| \leq |S| - 2.$$

Therefore,

$$|S| - 2 \leq \deg(0, y_1) \leq |S| - |\mathfrak{m}| \leq |S| - 2,$$

which implies that $\deg(0, y_1) = |S| - 2$ and $|\mathfrak{m}| = 2$. Now, $\deg(0, y_1) = |S| - 2$ implies that $(0, y_1)$ is not adjacent to $(1, y_i)$ for $i = 2, 3$. That is, for $i = 2, 3$,

$$(0, y_1) + (1, y_i) = (1, y_1 + y_i) \notin U(\mathbb{Z}_2 \times S)$$

and so $y_1 + y_i \notin U(S)$. This means that, by Lemma 3.17, for $i = 2, 3$, we have $y_1 + y_i \in \mathfrak{m}$. Therefore, by using $y_2 \neq y_3$ and $|\mathfrak{m}| = 2$, we may conclude that $\mathfrak{m} = \{y_1 + y_2, y_1 + y_3\}$. Without loss of generality, we may suppose that $y_1 + y_2 = 0$

and therefore $D = \{(0, y_1), (1, -y_1), (1, y_3)\}$. Since $y_1 + y_3 \in \mathfrak{m}$, we conclude that $-y_3 - y_1 \in \mathfrak{m}$ and so $-y_3 - y_1 \notin U(S)$. Therefore,

$$(0, -y_3) + (0, y_1) = (0, -y_3 + y_1) \notin U(\mathbb{Z}_2 \times S),$$
$$(0, -y_3) + (1, -y_1) = (1, -y_3 - y_1) \notin U(\mathbb{Z}_2 \times S) \text{ and}$$
$$(0, -y_3) + (1, y_3) = (1, 0) \notin U(\mathbb{Z}_2 \times S),$$

and so $(0, -y_3)$ is not adjacent to the elements of D. This implies that $(0, -y_3) \in D$ and therefore $(0, -y_3) = (0, y_1)$. This means that $y_1 + y_3 = 0$, which is a contradiction. This shows that $\gamma(G(\mathbb{Z}_2 \times S)) \geq 4$.

Now, by Lemma 4.1, we have $4 \leq \gamma(G(\mathbb{Z}_2 \times S)) \leq \gamma_t(G(\mathbb{Z}_2 \times S)) \leq 4$ and so $\gamma(G(\mathbb{Z}_2 \times S)) = \gamma_t(G(\mathbb{Z}_2 \times S)) = 4$, as required. □

Lemma 4.4. *Let R be a ring such that $R \cong R_1 \times R_2$, where R_1 and R_2 are local rings with maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. Suppose that either $|R_1/\mathfrak{m}_1| = 2$ or $|R_2/\mathfrak{m}_2| = 2$. If $R \ncong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, then $\gamma(G(R)) = \gamma_t(G(R)) = 4$.*

Proof. Without loss of generality, let $|R_1/\mathfrak{m}_1| = 2$. If $\mathfrak{m}_1 \neq \{0\}$, then R_1 is not a field. Therefore, by Lemma 4.2, $\gamma(G(R)) = \gamma_t(G(R)) = 4$. If $\mathfrak{m}_1 = \{0\}$, then $R_1 \cong \mathbb{Z}_2$ and since $R \ncong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, we conclude that R_2 is not a field. Now, Lemma 4.3 implies that $\gamma(G(R)) = \gamma_t(G(R)) = 4$. □

We also need Lemmas 4.5, 4.6 and 4.8 for proving part (3) of Theorem 2.2 and completing the paper.

Lemma 4.5. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two fields in which either both of them have characteristic 2 and none of them is isomorphic to $\mathbb{Z}_2$ or none of them has characteristic 2. Then $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) = \gamma_t(G(\mathbb{F}_1 \times \mathbb{F}_2)) = 3$.*

Proof. The assumption implies that $|\mathbb{F}_1| \geq 3$ and $|\mathbb{F}_2| \geq 3$ and so we may choose nonzero elements $a, c \in \mathbb{F}_1$ with $a \neq c$ and $b, d \in \mathbb{F}_2$ with $b \neq d$. We claim that $D = \{(0, 0), (a, b), (c, d)\}$ is a total dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$. In order to prove the claim, note that a given element in $\mathbb{F}_1 \times \mathbb{F}_2$ has one of the following forms, where $x \in \mathbb{F}_1$ and $y \in \mathbb{F}_2$ are nonzero elements: either $(0, 0)$, $(x, 0)$, $(0, y)$ or (x, y).

For $(0, 0)$, we have

$$(0, 0) + (a, b) = (a, b) \in U(\mathbb{F}_1 \times \mathbb{F}_2)$$

and so $(0, 0)$ is adjacent to (a, b).

For $(x, 0)$, we have either $x \neq -a$ or $x \neq -c$, and so either

$$(x, 0) + (a, b) = (x + a, b) \in U(\mathbb{F}_1 \times \mathbb{F}_2) \text{ or}$$
$$(x, 0) + (c, d) = (x + c, d) \in U(\mathbb{F}_1 \times \mathbb{F}_2).$$

This implies that $(x, 0)$ is adjacent to (a, b) or (c, d). By a similar argument, we obtain that $(0, y)$ is also adjacent to (a, b) or (c, d).

For (x, y), we have

$$(x, y) + (0, 0) = (x, y) \in U(\mathbb{F}_1 \times \mathbb{F}_2)$$

and so (x, y) is adjacent to $(0, 0)$.

Therefore, the claim holds true and so $\gamma_t(G(\mathbb{F}_1 \times \mathbb{F}_2)) \leq 3$. On the other hand, by Lemmas 3.4 and 3.6, we conclude that $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \geq 3$. Therefore,

$$3 \leq \gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) \leq \gamma_t(G(\mathbb{F}_1 \times \mathbb{F}_2)) \leq 3$$

and so $\gamma(G(\mathbb{F}_1 \times \mathbb{F}_2)) = \gamma_t(G(\mathbb{F}_1 \times \mathbb{F}_2)) = 3$, as required. □

Lemma 4.6. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two fields in which only one of them has characteristic 2 and none of them is isomorphic to $\mathbb{Z}_2$. Then $\gamma_t(G(\mathbb{F}_1 \times \mathbb{F}_2)) \leq 3$.*

Proof. Without loss of generality, we may assume that $\mathrm{Char}(\mathbb{F}_1) = 2$ and $\mathrm{Char}(\mathbb{F}_2) \neq 2$. Since $\mathbb{F}_1 \not\cong \mathbb{Z}_2$ and $\mathbb{F}_2 \not\cong \mathbb{Z}_2$, we conclude that $|\mathbb{F}_1| \geq 3$ and $|\mathbb{F}_2| \geq 3$, and so we may choose $a \in \mathbb{F}_1$ and $b \in \mathbb{F}_2$ such that $a \neq 0, 1$ and $b \neq -1, 1$. We claim that $D = \{(0, 1), (1, -1), (a, b)\}$ is a total dominating set of the unit graph $G(\mathbb{F}_1 \times \mathbb{F}_2)$. In order to prove the claim, note that a given element in $\mathbb{F}_1 \times \mathbb{F}_2$ has one of the following forms, where $x \in \mathbb{F}_1$ and $y \in \mathbb{F}_2$ are nonzero elements: either $(0, 0)$, $(x, 0)$, $(0, y)$ or (x, y).

For $(0, 0)$,

$$(0, 0) + (1, -1) = (1, -1) \in U(\mathbb{F}_1 \times \mathbb{F}_2)$$

implies that $(0, 0)$ is adjacent to $(1, -1)$.

For $(x, 0)$,

$$(x, 0) + (0, 1) = (x, 1) \in U(\mathbb{F}_1 \times \mathbb{F}_2)$$

implies that $(x, 0)$ is adjacent to $(0, 1)$.

For $(0, y)$, if $y \neq 1$, then we have

$$(0, y) + (1, -1) = (1, y - 1) \in U(\mathbb{F}_1 \times \mathbb{F}_2)$$

and if $y = 1$, then we have

$$(0, y) + (a, b) = (a, 1 + b) \in U(\mathbb{F}_1 \times \mathbb{F}_2).$$

Therefore, $(0, y)$ is either adjacent to $(1, -1)$ or (a, b).

Finally, for (x, y), if $x = y = 1$, then we have

$$(x, y) + (a, b) = (1 + a, 1 + b) \in U(\mathbb{F}_1 \times \mathbb{F}_2),$$

if $x \neq 1$ and $y = 1$, then we have

$$(x, y) + (0, 1) = (x, 2) \in U(\mathbb{F}_1 \times \mathbb{F}_2),$$

if $x = 1$ and $y \neq 1$, then for $y \neq -1$ we have

$$(x, y) + (0, 1) = (1, y + 1) \in U(\mathbb{F}_1 \times \mathbb{F}_2)$$

and for $y = -1$ we have

$$(x, y) + (a, b) = (1 + a, -1 + b) \in U(\mathbb{F}_1 \times \mathbb{F}_2),$$

and if $x, y \neq 1$, then we have

$$(x, y) + (1, -1) = (x + 1, y - 1) \in U(\mathbb{F}_1 \times \mathbb{F}_2).$$

Therefore, (x, y) is either adjacent to $(0, 1)$, $(1, -1)$ or (a, b).

Hence, the claim holds true and so $\gamma_t(G(\mathbb{F}_1 \times \mathbb{F}_2)) \leq 3$, as required. □

We need the following proposition for the proof of Lemma 4.8.

Proposition 4.7 ([1], Lemma 2.7(a)). *Let R be a ring, $x, y \in R$ and suppose that $J(R)$ denotes the Jacobson radical of R. If $x + J(R)$ and $y + J(R)$ are adjacent in the unit graph $G(R/J(R))$, then every element of $x + J(R)$ is adjacent to every element of $y + J(R)$ in the unit graph $G(R)$.*

Lemma 4.8. *Let R be a ring. Then $\gamma_t(G(R)) \leq \gamma_t(G(R/J(R)))$, where $J(R)$ denotes the Jacobson radical of R.*

Proof. Let $\gamma_t(G(R/J(R))) = n$ and consider $D = \{x_1 + J(R), \ldots, x_n + J(R)\}$ as a γ_t-set of $G(R/J(R))$. We claim that $D' = \{x_1, \ldots, x_n\}$ is a total dominating set of $G(R)$. In order to prove the claim, let $y \in R$ be given. Therefore, $y + J(R) \in R/J(R)$ and so there exists $x_i + J(R) \in D$ such that $y + J(R)$ is adjacent to $x_i + J(R)$ in $G(R/J(R))$. Now, Proposition 4.7 implies that y is adjacent to x_i in $G(R)$ and so the claim holds true. This implies that $\gamma_t(G(R)) \leq |D'| = n = \gamma_t(G(R/J(R)))$. □

We are now ready to prove part (3) of Theorem 2.2, that is:

Theorem 4.9. *Let R be a ring. Then $\gamma(G(R)) = 3$ if and only if R is not isomorphic to the product of two fields such that only one of them has characteristic 2, and $R \cong R_1 \times R_2$, where for $i = 1, 2$, R_i is a local ring with maximal ideal $\mathfrak{m}_i$ in such a way that $R_i/\mathfrak{m}_i \not\cong \mathbb{Z}_2$.*

Proof. ($\Rightarrow$): Let $\gamma(G(R)) = 3$. Note that, by Theorem 3.20, R is not isomorphic to the product of two fields such that only one of them has characteristic 2. Now, by Remark 3.15, we may write $R \cong R_1 \times \cdots \times R_n$, where for every $1 \leq i \leq n$, R_i is a local ring with maximal ideal $\mathfrak{m}_i$. If $n \geq 3$, then by using Lemma 3.16, we conclude that $3 = \gamma(G(R)) \geq n + 1 \geq 4$, which is a contradiction. Therefore, $n \leq 2$. If $n = 1$, then $R \cong R_1$ is either a field or a local ring which is not a field. In the first case, by Lemma 3.1, we have $\gamma(G(R)) = 1$, and in the second case, by Theorem 3.20, we have $\gamma(G(R)) = 2$, which both of them are contradictions. Therefore, $n = 2$ and so we have $R \cong R_1 \times R_2$. By Theorem 3.20, we conclude that $R \not\cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field, and so Lemma 4.4 implies that $|R_1/\mathfrak{m}_1| \geq 3$ and $|R_2/\mathfrak{m}_2| \geq 3$. Therefore, $R_1/\mathfrak{m}_1 \not\cong \mathbb{Z}_2$ and $R_2/\mathfrak{m}_2 \not\cong \mathbb{Z}_2$.

($\Leftarrow$): By the assumption and Remark 3.15, we have $R/J(R) \cong \mathbb{F}_1 \times \mathbb{F}_2$, where $\mathbb{F}_1 = R_1/\mathfrak{m}_1$ and $\mathbb{F}_2 = R_2/\mathfrak{m}_2$ are fields. Also, $\mathbb{F}_1$ and $\mathbb{F}_2$ are not isomorphic to $\mathbb{Z}_2$. Now, one of the following cases occurs:

Case 1: Either both of $\mathbb{F}_1$ and $\mathbb{F}_2$ have characteristic 2 or none of them has characteristic 2. In this case, by Lemma 4.5, we conclude that $\gamma(G(R/J(R))) = \gamma_t(G(R/J(R))) = 3$. Now, in view of Lemmas 3.14 and 4.8, we conclude that $3 \leq \gamma(G(R)) \leq \gamma_t(G(R)) \leq 3$, which implies that $\gamma(G(R)) = 3$, as required.

Case 2: Only one of $\mathbb{F}_1$ or $\mathbb{F}_2$ has characteristic 2. In this case, by Theorem 3.20, we have $\gamma(G(R/J(R))) = 2$, and also by Lemma 4.6, we have $\gamma_t(G(R/J(R))) \leq 3$. Therefore, in view of Lemmas 3.14 and 4.8, we conclude that either $\gamma(G(R)) = 2$ or $\gamma(G(R)) = 3$. Since $R \cong R_1 \times R_2$, R is not a local ring. Also, by the assumption, R is not isomorphic to the product of two fields such that only one of them has characteristic 2. Finally, in contrary, suppose that $R \cong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field. Therefore, $R/J(R) \cong \mathbb{Z}_2 \times \mathbb{F}$ and so $\mathbb{Z}_2 \times \mathbb{F} \cong \mathbb{F}_1 \times \mathbb{F}_2$, which implies that $|\mathbb{Z}_2 \times \mathbb{F}| = |\mathbb{F}_1 \times \mathbb{F}_2|$. The assumption implies that $\mathrm{Char}(\mathbb{F}) = 2$ and so $|\mathbb{Z}_2 \times \mathbb{F}|$ is a power of 2, while, by the assumption in Case 2, $|\mathbb{F}_1 \times \mathbb{F}_2|$ is divisible by an odd prime. This contradiction shows that $R \ncong \mathbb{Z}_2 \times \mathbb{F}$, where $\mathbb{F}$ is a field. Hence, by Theorem 3.20, $\gamma(G(R)) = 2$ is impossible and so $\gamma(G(R)) = 3$, as required. □

Acknowledgment

The authors would like to thank the referee for a careful reading of the paper and for valuable comments.

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S. Kiani, Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

E-mail address: `s.kiani@srbiau.ac.ir`

H. R. Maimani, Mathematics Section, Department of Basic Sciences, Shahid Rajaee Teacher Training University, P.O. Box 16785-163, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.

E-mail address: `maimani@ipm.ir`

M. R. Pournaki, Department of Mathematical Sciences, Sharif University of Technology, P.O. Box 11155-9415, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.

E-mail address: `pournaki@ipm.ir`

URL: `http://math.ipm.ac.ir/~pournaki/`

S. Yassemi, School of Mathematics, Statistics and Computer Science, College of Science, University of Tehran, Tehran, Iran, and School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran.

E-mail address: `yassemi@ut.ac.ir`

URL: `http://math.ipm.ac.ir/~yassemi/`