NEW CLASSES OF SET-THEORETIC COMPLETE INTERSECTION MONOMIAL IDEALS

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Abstract. Let Δ be a simplicial complex and χ be an s-coloring of Δ. Biermann and Van Tuyl have introduced the simplicial complex Δ_χ. As a corollary of Theorems 5 and 7 in their 2013 article, we obtain that the Stanley–Reisner ring of Δ_χ over a field is Cohen–Macaulay. In this note, we generalize this corollary by proving that the Stanley–Reisner ideal of Δ_χ over a field is set-theoretic complete intersection. This also generalizes a result of Macchia.

1. Statement of the main theorem

Let us start this note with some notions of combinatorics. A *simplicial complex* Δ on the set of *vertices* $V = \{v_1, \dots, v_n\}$ is a collection of subsets of V which is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ, and a *facet* of Δ is a maximal face of Δ with respect to inclusion. It is clear that all facets of Δ determines it. When $F_1, \dots, F_t$ are all facets of Δ, we write $\Delta = \langle F_1, \dots, F_t \rangle$. In this case, we say χ is an *s-coloring* of Δ when χ is a partition of V, say $V = W_1 \cup \cdots \cup W_s$ (where the sets W_j are allowed to be empty), such that for every $1 \leq i \leq t$ and every $1 \leq j \leq s$ the inequality $|F_i \cap W_j| \leq 1$ holds true. Biermann and Van Tuyl [3] have defined a new simplicial complex Δ_χ on the set of vertices $\{v_1, \dots, v_n, w_1, \dots, w_s\}$ with faces $\sigma \cup \tau$, where σ is a face of Δ and τ is any subset of $\{w_1, \dots, w_s\}$ such that for every $w_j \in \tau$, we have $\sigma \cap W_j = \emptyset$. It is shown in [3] that the facets of Δ_χ are in the form $F \cup F'$, where F is any face of Δ and $F' = \{w_i \mid F \cap W_i = \emptyset\}$. The simplicial complex Δ_χ is generally larger than Δ. For example, if $\Delta = \langle \{v_1, v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5, v_6\} \rangle$ is the simplicial complex on the set of vertices $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and χ is the 3-coloring of Δ given by $V = W_1 \cup W_2 \cup W_3$, where $W_1 = \{v_1, v_4\}$, $W_2 = \{v_2, v_5\}$, and $W_3 = \{v_3, v_6\}$, then we have

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$$\begin{aligned}\Delta_\chi = \ &\big\langle \{w_1, w_2, w_3\}, \{v_1, w_2, w_3\}, \{v_2, w_1, w_3\}, \{v_3, w_1, w_2\},\\ &\{v_4, w_2, w_3\}, \{v_5, w_1, w_3\}, \{v_6, w_1, w_2\}, \{v_1, v_2, w_3\},\\ &\{v_1, v_3, w_2\}, \{v_2, v_3, w_1\}, \{v_3, v_4, w_2\}, \{v_4, v_5, w_3\},\\ &\{v_4, v_6, w_2\}, \{v_5, v_6, w_1\}, \{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\big\rangle.\end{aligned}$$

Although many properties of the simplicial complex Δ_χ are known, it might be complicated in some sense comparing with Δ.

We now switch to the connection of the above-mentioned notions with commutative algebra. Let Δ be a simplicial complex on the set of vertices $V = \{v_1, \ldots, v_n\}$. The *Stanley–Reisner ideal* of Δ over a field $\mathbb{K}$ is the ideal I_Δ of $R = \mathbb{K}[x_1, \ldots, x_n]$, the polynomial ring in n variables over $\mathbb{K}$, which is defined as follows:

$$I_\Delta = \Big\langle \prod_{v_i \in F} x_i \mid F \notin \Delta \Big\rangle.$$

Also, the *Stanley–Reisner ring* of Δ over $\mathbb{K}$ is defined as R/I_Δ. One may ask about the arithmetical rank of the Stanley–Reisner ideal of a simplicial complex and whether it being set-theoretic complete intersection. We recall that for a given unitary commutative Noetherian ring R and for an ideal I of R, the *arithmetical rank* of I is denoted by ara(I) and is defined as the smallest integer t for which there exist $a_1, \ldots, a_t \in R$ such that $\sqrt{\langle a_1, \ldots, a_t\rangle} = \sqrt{I}$. The inequality $\mathrm{ht}(I) \leq \mathrm{ara}(I)$ holds true generally and in the case of equality, I is called *set-theoretic complete intersection.* The arithmetical rank of monomial ideals of $R = \mathbb{K}[x_1, \ldots, x_n]$ has been studied by several authors (see, for example, [1, 2, 6, 8, 9]).

We are now in the position to state the main theorem of this note. In the next section, we give a proof of the main theorem.

Main Theorem. *Let Δ be a simplicial complex and χ be an s-coloring of Δ. Then the Stanley–Reisner ideal of Δ_χ over a field is set-theoretic complete intersection.*

Before closing this section, we would like to point out that our main theorem generalizes two results of [3] and [9]. Let us explain some more things in detail to see this.

The main theorem as a generalization of a result of [3]**.** Let $R = \mathbb{K}[x_1, \ldots, x_n]$ and I be a squarefree monomial ideal of R. Lyubeznik [8] has proven that $\mathrm{pd}_R(R/I) \leq \mathrm{ara}(I)$, where $\mathrm{pd}_R(R/I)$ denotes the projective dimension of R/I. Thus

$$\mathrm{ht}(I) \leq \mathrm{pd}_R(R/I) \leq \mathrm{ara}(I).$$

This implies that, if I is set-theoretic complete intersection, then R/I is Cohen–Macaulay. Combining the above observation with our main theorem gives the following corollary, which is also a corollary of Theorems 5 and 7 in the article [3] of Biermann and Van Tuyl.

Corollary I ([3], Theorems 5 and 7)**.** *Let Δ be a simplicial complex and χ be an s-coloring of Δ. Then the Stanley–Reisner ring of Δ_χ over a field is Cohen–Macaulay.*

The main theorem as a generalization of a result of [9]**.** Let G be a finite simple graph with the set of vertices $V(G) = \{v_1, \dots, v_n\}$. To the graph G, one can associate an ideal $I(G) \subseteq \mathbb{K}[x_1, \dots, x_n]$ generated by all monomials $x_i x_j$ such that v_i and v_j are adjacent. The ideal $I(G)$ is called the *edge ideal* of G over $\mathbb{K}$. We recall that $A \subseteq V(G)$ is an *independent set* in G if none of its elements are adjacent. Based on this notion, the *independence simplicial complex* of G is defined by

$$\Delta_G = \big\{A \subseteq V(G) \mid A \text{ is an independent set in } G\big\}.$$

Note that Δ_G is precisely the simplicial complex with the Stanley–Reisner ideal $I(G)$. Cook and Nagel [4] have defined the fully clique-whiskered graphs in the following way. For a given graph G, $C \subseteq V(G)$ is called a *clique* of G if every pair of vertices of C are adjacent in G. Let χ be a partition of $V(G)$, say $V(G) = W_1 \cup \cdots \cup W_s$, such that W_i is a clique of G for every $1 \leq i \leq s$. Add new vertices $y_1, \dots, y_s$ and new edges vy_i for every $v \in W_i$ and every $1 \leq i \leq s$. The resulting graph is called a *fully clique-whiskered graph* of G. One can easily see that the independence simplicial complex of this latter graph is Δ_χ and so our main theorem gives the following corollary, which is Proposition 4.1 in the article [9] of Macchia.

Corollary II ([9], Proposition 4.1)**.** *Let G be a finite simple graph. Then the edge ideals of fully clique-whiskered graphs of G over a field are all set-theoretic complete intersections.*

2. Proof of the main theorem

We are now ready to prove the main theorem. In order to do this, suppose that the set of vertices of Δ is $V = \{v_1, \dots, v_n\}$ and $V = W_1 \cup \cdots \cup W_s$ is the s-coloring of Δ given by χ. Also, let $R = \mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field $\mathbb{K}$. Suppose that $\{v_1, \dots, v_n, w_1, \dots, w_s\}$ is the set of vertices of Δ_χ, and consider the Stanley–Reisner ideal I_{Δ_χ} of Δ_χ over $\mathbb{K}$ as a monomial ideal in the polynomial ring $R' = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_s]$.

Lemma A. *We have* $\operatorname{ht}(I_{\Delta_\chi}) = n$.

Proof. One can easily see that

$$I_{\Delta_\chi} = I_\Delta + \langle x_i y_j \mid v_i \in W_j,\ 1 \leq j \leq s\rangle.$$

Note that the ideal $\langle x_1, \dots, x_n\rangle$ is a minimal prime ideal of I_{Δ_χ}. Now [3, Theorem 5] implies that $\operatorname{ht}(I_{\Delta_\chi}) = n$, as required. □

Let us introduce the Lyubeznik resolution, which is defined in [7]. For every monomial ideal I of a polynomial ring R, there is an explicit free resolution of R/I, which is defined as follows. Assume that $G(I) = \{m_1, \dots, m_p\}$ is the minimal set of monomial generators of I. For every integer $t \geq 1$ and every sequence $1 \leq \ell_1 < \cdots < \ell_t \leq p$, consider the symbol $e_{\ell_1\cdots\ell_t}$. Let T_t be the free R-module generated by all symbols $e_{\ell_1\cdots\ell_t}$, $1 \leq \ell_1 < \cdots < \ell_t \leq p$. The differential map d_t is also defined by

$$d_t(e_{\ell_1\cdots\ell_t}) = \sum_{k=1}^{t}(-1)^{k-1}\frac{\mathrm{lcm}(m_{\ell_1},\ldots,m_{\ell_t})}{\mathrm{lcm}(m_{\ell_1},\ldots,\widehat{m_{\ell_k}},\ldots,m_{\ell_t})}e_{\ell_1\ldots\widehat{\ell_k}\ldots\ell_t}.$$

Then the complex $(T_\bullet, d_\bullet)$ is a free resolution of R/I and is called the *Taylor resolution* of I (see, for example, [5, Theorem 7.1.1]). Now fix an order $m_1 > \cdots > m_p$ on the set of minimal monomial generators of I. The symbol $e_{\ell_1\cdots\ell_t}$ is called *L-admissible* if m_q does not divide $\mathrm{lcm}(m_{\ell_k},\ldots,m_{\ell_t})$ for every $1 \leq k < t$ and every $1 \leq q < \ell_k$. Then the *Lyubeznik resolution* of I with respect to the above order of the minimal monomial generators is a subcomplex of the Taylor resolution generated by all L-admissible symbols. Since being L-admissible depends on an order of minimal monomial generators of I, a Lyubeznik resolution also depends on it. Finally, we say that an L-admissible symbol $e_{\ell_1\cdots\ell_t}$ is *maximal* if $e_{k_1\cdots k_{t'}}$ is not L-admissible whenever $\{\ell_1,\ldots,\ell_t\} \subsetneq \{k_1,\ldots,k_{t'}\}$.

Lemma B. *There is a Lyubeznik resolution of I_{Δ_χ} with length $\leq n$.*

Proof. Let $>$ be the pure lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_s$ (see [5, Example 2.1.2(c)]), and consider the Lyubeznik resolution of I_{Δ_χ} induced by $>$. We show that this specific Lyubeznik resolution of I_{Δ_χ} has length $\leq n$. In order to do this, let $m_1 > \cdots > m_p$ be the minimal monomial generators of I_{Δ_χ} and $e_{\ell_1\cdots\ell_k}$ be a maximal L-admissible symbol. We use induction on n to show that $k \leq n$. This completes the proof of the lemma.

One can easily check that $k \leq n$ is true for $n = 1, 2$. Thus we may assume that $n \geq 3$. Note that there exists a unique integer, say i_0, with $1 \leq i_0 \leq s$, such that $v_1 \in W_{i_0}$. We claim that there exists an integer $1 \leq i \leq n$ such that x_i divides m_{ℓ_1} but does not divide $\mathrm{lcm}(m_{\ell_2},\ldots,m_{\ell_k})$. In order to prove the claim, since $e_{\ell_1\ldots\ell_k}$ is a maximal L-admissible symbol, by using [6, Lemma 3], it follows that $\ell_1 = 1$. This implies that x_1 divides m_{ℓ_1}. Now, by the definition of L-admissible symbol, we conclude that there is a variable which divides m_{ℓ_1} but does not divide $\mathrm{lcm}(m_{\ell_2},\ldots,m_{\ell_k})$. If all of the variables which divide m_{ℓ_1} belong to the set $\{x_1,\ldots,x_n\}$, then we are done. Otherwise, we have $m_{\ell_1} = x_1 y_{i_0}$ since $x_1 \mid m_{\ell_1}$ and $v_1 \in W_{i_0}$. By the choice of $>$, we conclude that x_1 does not divide $\mathrm{lcm}(m_{\ell_2},\ldots,m_{\ell_k})$, and so the claim holds true. By the claim, we may assume that x_i divides m_{ℓ_1} but does not divide $\mathrm{lcm}(m_{\ell_2},\ldots,m_{\ell_k})$. Let

$$\Delta' = \Delta \setminus \{v_i\} = \{F \in \Delta \mid v_i \notin F\}$$

be a simplicial complex on the set of vertices $V' = V \setminus \{v_i\}$ and χ' be an s-coloring of Δ' given by $V' = W'_1 \cup \cdots \cup W'_s$, where $W'_j = W_j \setminus \{v_i\}$ for every $1 \leq j \leq s$. It is clear that $m_{\ell_2},\ldots,m_{\ell_k}$ belong to the set of minimal monomial generators of $I_{\Delta'_{\chi'}}$ and they determine an L-admissible symbol of $I_{\Delta'_{\chi'}}$. Since Δ' has $n-1$ vertices, our induction hypothesis implies that $k-1 \leq n-1$ and thus $k \leq n$. □

We also need to use the following known result of Kimura.

Lemma C ([6], Theorem 1). *Let I be a monomial ideal of a polynomial ring and λ be the minimum length of the Lyubeznik resolutions of I. Then* $\mathrm{ara}(I) \leq \lambda$.

Now, by combining Lemmas B and C, we get $\mathrm{ara}(I_{\Delta_\chi}) \leq n$. Therefore, Lemma A implies that

$$n = \mathrm{ht}(I_{\Delta_\chi}) \leq \mathrm{pd}_R(R/I_{\Delta_\chi}) \leq \mathrm{ara}(I_{\Delta_\chi}) \leq n,$$

and so $\mathrm{ht}(I_{\Delta_\chi}) = \mathrm{ara}(I_{\Delta_\chi}) = n$. This means that I_{Δ_χ} is a set-theoretic complete intersection ideal of R' and completes the proof of the main theorem. □

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