THE NONORIENTABLE GENUS OF SOME JACOBSON GRAPHS AND CLASSIFICATION OF THE PROJECTIVE ONES

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Abstract. Let R be a finite commutative ring with nonzero identity and denote its Jacobson radical by $J(R)$. The Jacobson graph of R is the graph in which the vertex set is $R \setminus J(R)$, and two distinct vertices x and y are adjacent if and only if $1 - xy$ is not a unit in R. In this paper, the nonorientable genus of some Jacobson graphs is either computed or estimated by a lower bound. As an application, the rings R with projective Jacobson graphs are classified.

1. Introduction

A *surface* is a real 2-manifold, that is, a connected Hausdorff topological space with a countable basis such that every point lies in an open neighborhood homeomorphic to an open neighborhood of $\mathbb{R}^2$. A *nonorientable surface* is a surface that contains a subspace which is homeomorphic to a Möbius strip. A *connected sum* of two surfaces is the surface obtained by cutting an open disc out of each surface and then gluing the two surfaces together along the boundaries of the holes. For the positive integer q, N_q denotes the surface formed by a connected sum of q projective planes. It is easy to see that N_q is a nonorientable surface. We may denote the sphere by N_0, even though it is orientable. The surface N_1 is the projective plane and the surface N_2 is the Klein bottle. The nonnegative integer q is called the *nonorientable genus* of N_q. The *nonorientable genus* of a finite graph G without loops or multiple edges, denoted by $\bar{\gamma}(G)$, is the smallest nonnegative integer q for which there exists an embedding of G into N_q. The graphs G with $\bar{\gamma}(G) = 0$ are in fact planar graphs and the graphs G with $\bar{\gamma}(G) = 1$ are called *projective graphs*. We refer the reader to [8] for a treatment of the subject.

In this paper, we deal with some graphs arising from rings. These graphs are called Jacobson graphs. We compute the nonorientable genus of some of them and give a lower bound for the some other. As an application, we classify, up to isomorphism, the finite commutative rings with nonzero identity whose Jacobson graphs are projective. It is noticeable that associating a graph to an algebraic structure is a research subject in the area of algebraic combinatorics and has attracted considerable attention. The research in this subject aims at exposing the relationship between algebra and graph theory and at advancing the application of one to the other. In fact, there are three

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major problems in this subject: (1) characterization of the resulting graphs, (2) characterization of the algebraic structures with isomorphic graphs, and (3) realization of the connections between the algebraic structures and the corresponding graphs. In literature, one can find a number of different types of graphs attached to rings or other algebraic structures. We refer the reader to [2] and [7] for a survey of recent results concerning graphs attached to rings.

2. Preliminaries

In this section, we recall some definitions and notations concerning graphs for later use. In the sequel, by a graph we mean a finite graph without loops or multiple edges. Also, for the convenience of the reader, we state without proof a few known results in the form of propositions which will be used later. For unexplained terminology and notations concerning graph theory we refer the reader to [4].

The present paper deals with what is known as the Jacobson graph of a ring. Some of the properties of this graph have been studied in detail in [3]. First, let us state the definition of Jacobson graphs.

Definition 2.1. Let R be a finite commutative ring with nonzero identity, $J(R)$ denote the Jacobson radical of R and let $U(R)$ be the set of unit elements in R. The *Jacobson graph* of R, denoted by $\mathfrak{J}_R$, is defined to be the graph in which the vertex set is $R\setminus J(R)$, and two distinct vertices x and y are adjacent if and only if $1-xy \notin U(R)$.

The graphs in Figure 1 are the Jacobson graphs of the rings indicated. Throughout the paper, in all the figures, we abbreviate the ordered pair (r, s) by rs.

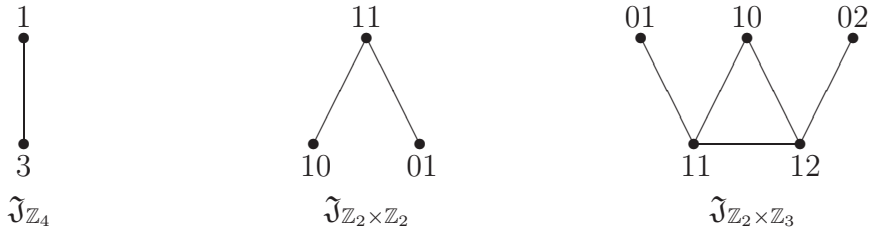

Figure 1. The Jacobson graphs of some specific rings.

The following result of paper [3] characterizes the planar Jacobson graphs which will be used later frequently. Note that the ring $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$ is missed in the paper [3] and $\mathbb{F}_4$ denotes the ring $\frac{\mathbb{Z}_2[x]}{(x^2+x+1)}$.

Proposition 2.2 ([3], Theorem 4.3)**.** *Let R be a finite commutative ring with nonzero identity. Then the Jacobson graph $\mathfrak{J}_R$ is a planar graph if and only if either R is a field or it is isomorphic to one of $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_3$, $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{F}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_3\times\mathbb{Z}_3$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_3[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$ or $\frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$.*

We now state the following proposition which is needed for later use. Let us recall that a graph G is said to be *complete* if there is an edge between every pair of distinct

vertices in G. We denote the complete graph with n vertices by K_n. A *bipartite graph* is the one whose vertex set can be partitioned into two disjoint parts in such a way that the two end vertices of every edge lie in different parts. Among the bipartite graphs, the *complete bipartite graph* is the one in which two vertices are adjacent if and only if they lie in different parts. The complete bipartite graph, with parts of size m and n, is denoted by $K_{m,n}$.

Proposition 2.3 ([3], Theorem 2.2)**.** *Let R be a finite commutative ring with nonzero identity. If R is local with maximal ideal $\mathfrak{m}$ and residue field $\mathbb{K} = R/\mathfrak{m}$, then the connected components of $\mathfrak{J}_R$ are isomorphic either to complete graph $K_{|\mathfrak{m}|}$ or to complete bipartite graph $K_{|\mathfrak{m}|,|\mathfrak{m}|}$. Moreover,*

(1) *if $|\mathbb{K}|$ is odd, then $\mathfrak{J}_R$ has two complete components and $(|\mathbb{K}|-3)/2$ complete bipartite components, and*

(2) *if $|\mathbb{K}|$ is even, then $\mathfrak{J}_R$ has one complete component and $(|\mathbb{K}|-2)/2$ complete bipartite components.*

The following two propositions give us the nonorientable genus of the complete and complete bipartite graphs.

Proposition 2.4 ([10], Theorem 11–19)**.** *If $n \geq 3$, then*

$$\bar{\gamma}(K_n) = \begin{cases} \left\lceil \dfrac{(n-3)(n-4)}{6} \right\rceil & \text{if } n \neq 7, \\ 3 & \text{if } n = 7. \end{cases}$$

Proposition 2.5 ([10], Theorem 11–23)**.** *If $m, n \geq 2$, then*

$$\bar{\gamma}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil.$$

The following two propositions give us a lower bound for the nonorientable genus of connected graphs.

Proposition 2.6 ([10], Theorem 11–7)**.** *If G is a connected graph with $n \geq 3$ vertices and m edges, then $\bar{\gamma}(G) \geq \frac{m}{3} - n + 2$.*

Proposition 2.7 ([10], Theorem 11–8)**.** *If G is a connected graph with $n \geq 3$ vertices and m edges and has no triangles, then $\bar{\gamma}(G) \geq \frac{m}{2} - n + 2$.*

Given a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. If G_1 and G_2 are any two graphs, then their *disjoint union*, denoted by $G_1 \sqcup G_2$, is defined to be the graph in which the vertex set is $V(G_1) \sqcup V(G_2)$ and the edge set is $E(G_1) \sqcup E(G_2)$. The following result, which follows from [5, Theorem 1.4], often enables us to reformulate some results which are otherwise true for connected graphs. Let us recall that the *genus* of a graph G, denoted by $\gamma(G)$, is the smallest nonnegative integer g such that the graph G can be embedded on the surface obtained by attaching g handles to a sphere.

Proposition 2.8 ([5], Theorem 1.4)**.** *If a graph G is isomorphic to the disjoint union $G_1 \sqcup G_2$ of two graphs G_1 and G_2, then $\bar{\gamma}(G_1 \sqcup G_2) = \bar{\gamma}(G_1)+\bar{\gamma}(G_2)+\delta$, where $\delta = -1$ if either $\bar{\gamma}(G_1) > 2\gamma(G_1)$ or $\bar{\gamma}(G_2) > 2\gamma(G_2)$, and $\delta = 0$ otherwise.*

3. The Main Theorem

Rings whose Jacobson graphs are either planar or toroidal have been already classified (see [3, Theorem 4.3] and [1, Theorem 1.1]). As the main result of this paper, we classify, up to isomorphism, the rings whose Jacobson graphs are projective. We recall that, in the sequel, $\mathbb{F}_4$ denotes the ring $\frac{\mathbb{Z}_2[x]}{(x^2+x+1)}$.

Main Theorem. *Let R be a finite commutative ring with nonzero identity. Then the Jacobson graph $\mathfrak{J}_R$ is a projective graph if and only if R is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{F}_4$ or $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.*

We divide the proof of the Main Theorem into a series of lemmas. Let us start with the following one.

Lemma 3.1. *Let R be a finite commutative ring with nonzero identity having a maximal ideal of size ≥ 7. Then $\bar{\gamma}(\mathfrak{J}_R) \geq 3$.*

Proof. By the assumption, R has at least a maximal ideal $\mathfrak{m} = \{m_1, \ldots, m_\ell\}$ with $\ell \geq 7$. Let $i \neq j$ with $1 \leq i, j \leq \ell$ be given. Note that $1 + m_i$ and $1 + m_j$ are distinct elements of $R \setminus J(R)$, and $1 - (1+m_i)(1+m_j) \notin U(R)$. Therefore, $1+m_i$ and $1+m_j$ are adjacent vertices in $\mathfrak{J}_R$. This implies that the vertices $1 + m_1, \ldots, 1 + m_\ell$ are mutually adjacent in the Jacobson graph $\mathfrak{J}_R$ and so it has a subgraph isomorphic to K_ℓ, where $\ell \geq 7$. Now, Proposition 2.4 implies that $\bar{\gamma}(\mathfrak{J}_R) \geq 3$. □

We also need to characterize all finite commutative rings with nonzero identity which are either a local ring, a product of two local rings or a product of three local rings, and all of their maximal ideals have size ≤ 6. In the following lemma, we give this characterization.

Lemma 3.2. *Let R be a finite commutative ring with nonzero identity such that its all maximal ideals have size ≤ 6. Then the following statements hold true:*

(a) *If R is a local ring, then it is either a field or is isomorphic to one of the rings given by Table 1.*

(b) *If R is a product of two local rings, then it is isomorphic to one of the rings given by Table 2.*

(c) *If R is a product of three local rings, then it is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.*

Proof. (a) This part of the lemma may be obtained by using [6] together with some known results on the structures of small local rings.

(b) By the assumption, R is a product of two local rings, say R_1 and R_2. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be the maximal ideals of R_1 and R_2, respectively.

$\lvert\mathfrak{m}\rvert$	$\lvert R\rvert$	R
5	25	$\mathbb{Z}_{25},\ \frac{\mathbb{Z}_5[x]}{(x^2)}$
4	16	$\frac{\mathbb{Z}_4[x]}{(x^2+x+1)},\ \frac{\mathbb{F}_4[x]}{(x^2)}$
4	8	$\mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{(x^3)},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2)},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)},\ \frac{\mathbb{Z}_2[x,y]}{(x,y)^2}$
3	9	$\mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{(x^2)}$
2	4	$\mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{(x^2)}$

TABLE 1. Finite commutative local rings R with maximal ideals $\mathfrak{m}$ of size ≤ 6.

$\lvert\mathfrak{m}\rvert$	R
≤ 6	$\mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$
≤ 5	$\mathbb{Z}_2 \times \mathbb{Z}_5,\ \mathbb{Z}_3 \times \mathbb{Z}_5,\ \mathbb{F}_4 \times \mathbb{Z}_5,\ \mathbb{Z}_5 \times \mathbb{Z}_5$
≤ 4	$\mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{F}_4,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_3 \times \mathbb{F}_4,\ \mathbb{F}_4 \times \mathbb{F}_4$
≤ 3	$\mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{Z}_3$
≤ 2	$\mathbb{Z}_2 \times \mathbb{Z}_2$

TABLE 2. Finite commutative rings R with maximal ideals $\mathfrak{m}$ of size ≤ 6 which are a product of two local rings.

First, we may assume that none of R_1 and R_2 is a field. In this case, for $i = 1, 2$, we have $|R_i| \geq 4$ and $|\mathfrak{m}_i| \geq 2$. Note that $\mathfrak{m}_1 \times R_2$ is a maximal ideal of R with $|\mathfrak{m}_1 \times R_2| \geq 8$, a contradiction. Hence, this is not the case and so either both of R_1 and R_2 are fields or only one of them is a field.

Case 1: Both of R_1 and R_2 are fields. In this case, $\{0\} \times R_2$ and $R_1 \times \{0\}$ are the maximal ideals of R. Now, by the assumption, for $i = 1, 2$, we obtain that $|R_i| \leq 5$. Therefore, R is isomorphic to one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{F}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_3 \times \mathbb{F}_4,\ \mathbb{Z}_3 \times \mathbb{Z}_5,$$
$$\mathbb{F}_4 \times \mathbb{F}_4,\ \mathbb{F}_4 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_5 \times \mathbb{Z}_5.$$

Case 2: Only, say R_1, is a field. In this case, R_2 is not a field and so we have $|R_2| \geq 4$ and $|\mathfrak{m}_2| \geq 2$. Note that $\{0\} \times R_2$ and $R_1 \times \mathfrak{m}_2$ are the maximal ideals of R.

Now, by the assumption, $|R_1| \leq 3$ and $4 \leq |R_2| \leq 6$. Therefore, R is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_3 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.

(c) By the assumption, R is a product of three local rings, say R_1, R_2 and R_3. If one of the R_is has size ≥ 4, then R has a maximal ideal of size ≥ 8, a contradiction. If two of the R_is have size equal to 3, then R has a maximal ideal of size ≥ 9, again a contradiction. Therefore, R is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. □

Let us continue the process of classification by looking at some projective Jacobson graphs.

Lemma 3.3. *If R is one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{F}_4$ and $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, then $\bar{\gamma}(\mathfrak{J}_R) = 1$.*

Proof. By Proposition 2.2, we have $\bar{\gamma}(\mathfrak{J}_R) \geq 1$. On the other hand, for each of the given rings R in the statement of the lemma, one of the Figures 2, 3, 4 and 5 gives an embedding of the Jacobson graph $\mathfrak{J}_R$ on a projective plane. Hence, $\bar{\gamma}(\mathfrak{J}_R) = 1$. □

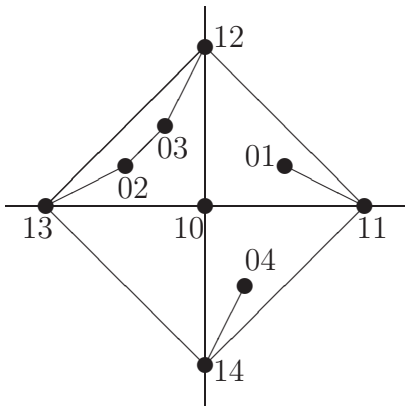

FIGURE 2. Embedding of the Jacobson graph of $\mathbb{Z}_2 \times \mathbb{Z}_5$ on a projective plane.

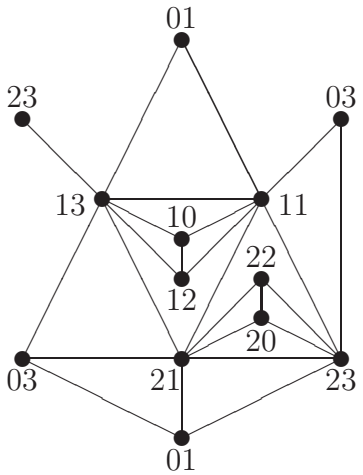

FIGURE 3. Embedding of the Jacobson graph of $\mathbb{Z}_3 \times \mathbb{Z}_4$ on a projective plane.

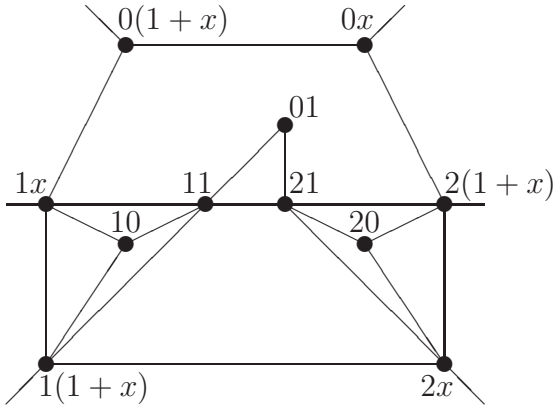

FIGURE 4. Embedding of the Jacobson graph of $\mathbb{Z}_3 \times \mathbb{F}_4$ on a projective plane.

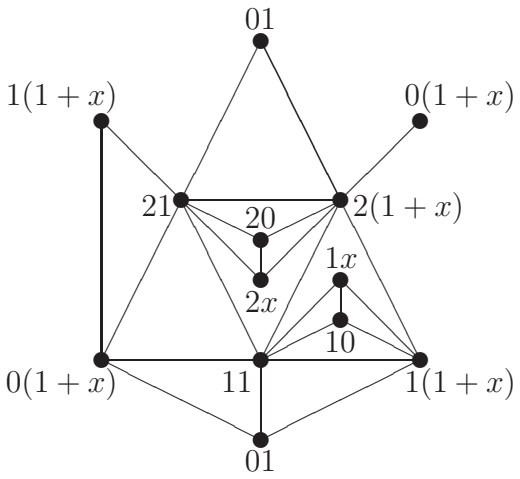

FIGURE 5. Embedding of the Jacobson graph of $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$ on a projective plane.

We continue the paper by looking at some rings whose Jacobson graphs have nonorientable genus greater than one. Let us start with the following lemma.

Lemma 3.4. *If R is one of the rings $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$ and $\frac{\mathbb{F}_4[x]}{(x^2)}$, then $\bar{\gamma}(\mathfrak{J}_R) = 2$.*

Proof. By Proposition 2.3, the Jacobson graph $\mathfrak{J}_R$ is isomorphic to the disjoint union of a copy of K_4 and a copy of $K_{4,4}$. Hence, by Propositions 2.8, 2.4 and 2.5, we obtain that $\bar{\gamma}(\mathfrak{J}_R) = 2$. □

Lemma 3.5. *If R is one of the rings $\mathbb{Z}_{25}$ and $\frac{\mathbb{Z}_5[x]}{(x^2)}$, then $\bar{\gamma}(\mathfrak{J}_R) = 7$.*

Proof. By Proposition 2.3, the Jacobson graph $\mathfrak{J}_R$ is isomorphic to the disjoint union of two copies of K_5 and a copy of $K_{5,5}$. Hence, by Propositions 2.8, 2.4 and 2.5, we obtain that $\bar{\gamma}(\mathfrak{J}_R) = 7$. □

Before stating the next lemma, we recall that for a graph G and a subset V of the vertex set of G, the subgraph of G whose vertex set is V and whose edge set consists of all edges in G that have both endpoints in V is called the *induced subgraph of G by V* and is denoted by $\langle V\rangle$.

Lemma 3.6. *If $R = \mathbb{Z}_3 \times \mathbb{Z}_5$, then $\bar{\gamma}(\mathfrak{J}_R) \geq 2$.*

Proof. Consider the following two subsets V_1 and V_2 of the vertex set of $\mathfrak{J}_R$:

$$V_1 = \{(1,0),(1,1),(1,2),(1,3),(1,4)\},$$
$$V_2 = \{(2,0),(2,1),(2,2),(2,3),(2,4)\}.$$

It is easy to see that the two subgraphs $\langle V_1\rangle$ and $\langle V_2\rangle$ are disjoint, and $\langle V_1\rangle \cong \langle V_2\rangle \cong K_5$. Therefore, it follows from Propositions 2.8 and 2.4 that $\bar{\gamma}(\mathfrak{J}_R) \geq 2$. □

Lemma 3.7. *If $R = \mathbb{F}_4 \times \mathbb{F}_4$, then $\bar{\gamma}(\mathfrak{J}_R) \geq 3$.*

Proof. Let G be a graph formed by removing the following 8 edges from the Jacobson graph $\mathfrak{J}_R$:

$$\{(0,1),(1,1)\},\ \{(1,1),(1,0)\},$$
$$\{(1,x),(1,x+1)\},\ \{(x,1),(x+1,1)\},$$
$$\{(x,x),(x+1,x)\},\ \{(x+1,x),(x+1,x+1)\},$$
$$\{(x+1,x+1),(x,x+1)\},\ \{(x,x+1),(x,x)\}.$$

It is easy to see that G is a connected graph with 15 vertices and 32 edges and has no triangles, so by Proposition 2.7, we obtain that $\bar{\gamma}(G) \geq 3$. This implies that $\bar{\gamma}(\mathfrak{J}_R) \geq 3$. □

Lemma 3.8. *If $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, then $\bar{\gamma}(\mathfrak{J}_R) \geq 3$.*

Proof. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_2 \times \mathbb{Z}_6$, it is enough to prove the lemma for $R = \mathbb{Z}_2 \times \mathbb{Z}_6$. It is easy to see that $\mathfrak{J}_R$ is a connected graph with 11 vertices and 35 edges, so by Proposition 2.6, we obtain that $\bar{\gamma}(\mathfrak{J}_R) \geq \frac{35}{3} - 11 + 2 > 2$. Therefore, $\bar{\gamma}(\mathfrak{J}_R) \geq 3$. □

Lemma 3.9. *If R is one of the rings $\mathbb{F}_4 \times \mathbb{Z}_5$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$, then $\bar{\gamma}(\mathfrak{J}_R) \geq 5$.*

Proof. Consider the following two subsets V_1 and V_2 of the vertex set of $\mathfrak{J}_R$. If $R = \mathbb{F}_4 \times \mathbb{Z}_5$ is the case, we may consider

$$V_1 = \{(x,0),(x,1),(x,2),(x,3),(x,4)\},$$
$$V_2 = \{(x+1,0),(x+1,1),(x+1,2),(x+1,3),(x+1,4)\},$$

and if $R = \mathbb{Z}_5 \times \mathbb{Z}_5$ is the case, we may consider

$$V_1 = \{(2,0),(2,1),(2,2),(2,3),(2,4)\},$$
$$V_2 = \{(3,0),(3,1),(3,2),(3,3),(3,4)\}.$$

In each cases, it is easy to see that $\langle V_1 \cup V_2\rangle$ has a subgraph isomorphic to $K_{5,5}$. Therefore, it follows from Propositions 2.8 and 2.5 that $\bar{\gamma}(\mathfrak{J}_R) \geq 5$. □

Now, by using Lemma 3.2 together Proposition 2.2 and all of the remaining lemmas (except Lemma 3.1) one has Table 3.

R	$\bar{\gamma}(\mathfrak{J}_R)$	type of R
$\mathbb{Z}_4,\ \mathbb{Z}_8,\ \mathbb{Z}_9,\ \frac{\mathbb{Z}_2[x]}{(x^2)},\ \frac{\mathbb{Z}_2[x]}{(x^3)},\ \frac{\mathbb{Z}_3[x]}{(x^2)},$ $\frac{\mathbb{Z}_4[x]}{(2x,x^2)},\ \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)},\ \frac{\mathbb{Z}_2[x,y]}{(x,y)^2},$ a field	0	a local ring
$\mathbb{Z}_2\times\mathbb{Z}_2,\ \mathbb{Z}_2\times\mathbb{Z}_3,\ \mathbb{Z}_2\times\mathbb{Z}_4,$ $\mathbb{Z}_2\times\mathbb{F}_4,\ \mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{(x^2)},\ \mathbb{Z}_3\times\mathbb{Z}_3$	0	a product of two local rings
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	0	a product of three local rings
$\mathbb{Z}_2\times\mathbb{Z}_5,\ \mathbb{Z}_3\times\mathbb{Z}_4,$ $\mathbb{Z}_3\times\mathbb{F}_4,\ \mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{(x^2)}$	1	a product of two local rings
$\frac{\mathbb{Z}_4[x]}{(x^2+x+1)},\ \frac{\mathbb{F}_4[x]}{(x^2)}$	2	a local ring
$\mathbb{Z}_{25},\ \frac{\mathbb{Z}_5[x]}{(x^2)}$	7	a local ring
$\mathbb{Z}_3\times\mathbb{Z}_5$	≥ 2	a product of two local rings
$\mathbb{F}_4\times\mathbb{F}_4$	≥ 3	a product of two local rings
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3$	≥ 3	a product of three local rings
$\mathbb{F}_4\times\mathbb{Z}_5,\ \mathbb{Z}_5\times\mathbb{Z}_5$	≥ 5	a product of two local rings

TABLE 3. Nonorientable genus of all of the Jacobson graphs arising from finite commutative rings R with nonzero identity which are either a local ring, a product of two local rings or a product of three local rings, and all of their maximal ideals have size ≤ 6.

We are now in the position to complete the proof of the Main Theorem.

Proof of the Main Theorem. ($\Rightarrow$): Suppose that R is a finite commutative ring with nonzero identity such that its Jacobson graph is projective, that is, $\bar{\gamma}(\mathfrak{J}_R)=1$. By [9, Page 95], we may write $R\cong R_1\times\cdots\times R_\ell$, where every R_i is a local ring with maximal ideal $\mathfrak{m}_i$. Since $\bar{\gamma}(\mathfrak{J}_R)=1$, by Lemma 3.1, the size of all of the maximal ideals of R is at most 6. This forces that $\ell\leq 3$, that is, either R is a local ring, R is a product of two local rings, or a product of three local rings. Now, Table 3 implies that R is isomorphic to one of $\mathbb{Z}_2\times\mathbb{Z}_5,\ \mathbb{Z}_3\times\mathbb{Z}_4,\ \mathbb{Z}_3\times\mathbb{F}_4$ or $\mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{(x^2)}$.

($\Leftarrow$): If R is isomorphic to one of $\mathbb{Z}_2\times\mathbb{Z}_5,\ \mathbb{Z}_3\times\mathbb{Z}_4,\ \mathbb{Z}_3\times\mathbb{F}_4$ or $\mathbb{Z}_3\times\frac{\mathbb{Z}_2[x]}{(x^2)}$, then again by Table 3 we obtain that $\bar{\gamma}(\mathfrak{J}_R)=1$. □

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